[redacted]

CALIFORNIA INSTITUTE OF TECHNOLOGY

DYNAMICS LABORATORY

NUMERICAL RESPONSE CHARACTERISTICS OF A UNIFORM BEAM CARRYING ONE DISCRETE LOAD

by

Loren D. Lutes

A report on research conducted for the National Aeronautics and Space Administration and the National Science Foundation

Pasadena, California

1965

Numerical Response Characteristics of a Uniform Beam Carrying One Discrete Load

by

Loren D. Lutes

A report issued under the project

"RESEARCH ON FAILURE OF EQUIPMENT WHEN SUBJECT TO VIBRATION"

for

Marshall Space Flight Center
National Aeronautics and Space Administration

under

Contract No. NAS8-2451

California Institute of Technology
Pasadena, California
July 21, 1965

Forward

One of the fundamental subjects studied under the project "Research on Failure of Equipment When Subject to Vibration" is the representation of complicated mechanical systems by relatively simple dynamic models. Such models must, of course, be sufficiently simple that mathematical computations are feasible, but at the same time they must be complicated enough to preserve the dynamic properties of real significance for a particular problem. In the course of these investigations it appeared that a tabulation of beam properties for a series of beams with discrete loads would be a useful aid to the design of experiments in the subject.

At the time of the death of Professor Charles E. Crede, Principal Investigator of the NAS8-2451 Project, calculations were well advanced towards a compilation of the basic dynamic characteristics of a series of loaded beams covering a number of practical situations. It was decided that in view of the potential value of such a collection of results for future studies in the field, the computations in hand should be completed and the results issued in report form.

I should like to express my appreciation to Mr. Loren D. Lutes for his efforts in completing this work and presenting the results in a compact and convenient form.

D. E. Hudson
Professor of Mechanical Engineering
and Applied Mechanics
California Institute of Technology

Abstract

Many practical cases of uniform beams carrying discrete loads cannot be accurately approximated either as single degree of freedom systems, or as simple uniform beams. A mathematical analysis of such systems is presented, where Bernoulli-Euler theory is used for the beam, and the effects of both the mass and the rotational inertia of the discrete load are included.

Numerical values of various normal mode factors are presented for seven particular systems - - namely: a fixed-free beam with the discrete load at the tip, and fixed-fixed and hinged-hinged beams with the discrete load at the mid-point, at the one-third point, and at the one-sixth point. These factors enable one to obtain natural frequencies, mode shapes, and modal response to support motion excitation.

Introduction. If an ideal elastic beam subjected to support motion carries a discrete mass which is large compared to the mass of the beam itself, then the beam can be approximated as a massless spring and the equation of motion for the discrete load can be written as

$$M\ddot{y} + Ky = -M\ddot{s} \tag{1}$$

where M is the mass of the discrete load, K is the stiffness constant of the beam (given by $K = \frac{Mg}{\text{static deflection}}$, g is the acceleration of gravity), y denotes motion relative to the support, and s denotes motion of the support. The general solution of this single degree of freedom problem is treated in any elementary vibration text. The stress in the beam for a given displacement of the discrete load can be approximated by assuming that the deflected shape of the beam during dynamic response is similar to the static deflection shape corresponding to the discrete load. One assumption which has been made above, but which is not always explicitly stated when this type approximation is employed, is that the kinetic energy of the discrete load due to rotation during osillation is small compared to that due to translation. This assumption is exact in cases where, because of symmetry, the discrete load does not rotate during oscillation, and the approximation is good if the radius of gyration of the discrete load in the plane of motion is very small compared to the length of the beam.

Another well known limiting case of the system under consideration is that of a uniform beam with no discrete load. For a slender beam one can neglect the effects of rotatory inertia and transverse shear deflection, and use the Bernoulli-Euler equation for an elastic beam:

$$EI \frac{\partial^4 y(x,t)}{\partial x^4} + \mu \frac{\partial^2 y(x,t)}{\partial t^2} = -\mu \ddot{s} \qquad (2)$$

where μ is the mass per unit length and EI the flexural rigidity of the beam. One of the most widely used methods of treating such a continuous system is the normal mode technique. This method has the advantage of reducing the problem to the solution of an infinite set of uncoupled equations, each of which is identical in form to the equation for a single degree of freedom system. Thus if the solution of Equation (1) for a given type of support motion is known, it is simple to calculate the solution of Equation (2) for this support motion, once the normal mode constants have been computed. The values of these normal mode constants for various common beam configurations have been published by several authors (ref. 1, 2, 3), and some of them are reproduced for reference in Table 3.

Unfortunately many practical cases of beams carrying discrete loads cannot be accurately approximated by either of the above simplified cases. The following is an application of the normal mode technique to such intermediate situations. The effect of the rotational inertia of the discrete load is included and proves to be significant for many cases.

References 4, 5, and 6 present some alternative methods which could be used to treat the problem considered here. Reference 7 extends the theory to include a beam carrying many discrete loads, but no numerical results are presented.

General Formulation. In the normal mode technique a set of orthogonal functions, $\varphi_k(x)$, are found, each of which satisfies the reduced equation of motion (i. e. $s \equiv 0$) and the boundary conditions. These orthogonal functions (called normal mode shapes) form a complete set of eigenfunctions for the system, and hence the response of the system to any excitation can be written as

$$y(x,t) = \sum_{k=1}^{\infty} \xi_k(t)\, \varphi_k(x) \quad . \tag{3}$$

The reduced equations of motion for the beam carrying a discrete load at the point $x = \ell_1$, with coordinates as shown in Figure 1, are:

$$\frac{\partial^4 y}{\partial x^4} + \frac{\mu}{EI}\frac{\partial^2 y}{\partial t^2} = 0 \qquad (x \neq \ell_1) \tag{4}$$

$$EI\,\Delta\left(\frac{\partial^3 y}{\partial x^3}\right) = -M\frac{\partial^2 y}{\partial t^2}(\ell_1) \tag{5a}$$

$$EI\,\Delta\left(\frac{\partial^2 y}{\partial x^2}\right) = M\rho^2\frac{\partial^3 y}{\partial x \partial t^2}(\ell_1) \tag{5b}$$

where $\Delta\left(\frac{\partial^n y}{\partial x^n}\right)$ is the finite increase in the n^{th} derivative at $x = \ell_1$, and $M\rho^2$ is the moment of inertia of the discrete load about the neutral plane of the beam. Boundary conditions at $x = 0$ and $x = \ell$ must also be satisfied.

In order to find a set of functions $\varphi_k(x)$, each of which satisfy Equations (4) and (5) one can assume that there exist solutions which can be written in the form

$$y = \psi_k(t)\, \varphi_k(x) \quad . \tag{6}$$

Substituting into Equation (4) gives

$$\frac{d^4\varphi_k}{dx^4}\,\psi_k + \frac{\mu}{EI}\,\varphi_k\,\frac{d^2\psi_k}{dt^2} = 0 \tag{7}$$

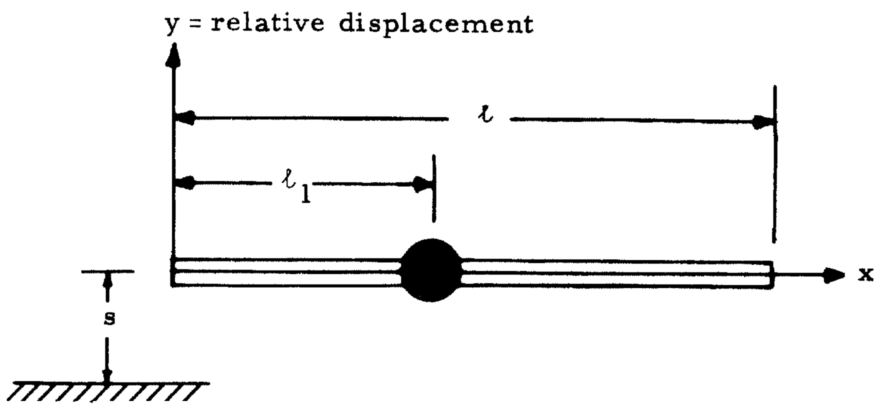


Figure 1. Uniform Beam with One Discrete Load.

or

$$\frac{\left(\frac{d^4\varphi_k}{dx^4}\right)}{\frac{\mu}{EI}\varphi_k} = -\frac{\left(\frac{d^2\psi_k}{dt^2}\right)}{\psi_k} = p_k^2 \tag{8}$$

where p_k is a constant. The form of Equation (8) shows that the free vibration of $\varphi_k(x)$ is harmonic with circular frequency p_k. Substitution of the fact that

$$\frac{d^2\psi_k}{dt^2} = -\ p_k^2\,\psi_k$$

from Equation (8) into Equations (4) and (5) gives the following relationships:

$$\frac{d^4\varphi_k}{dx^4} - \frac{\mu}{EI}\,p_k^2\varphi_k = 0 \qquad (x \neq \ell_1) \tag{9}$$

$$EI\,\Delta\left(\frac{d^3\varphi_k}{dx^3}\right) = M\,p_k^2\,\varphi_k(\ell_1) \tag{10a}$$

$$EI\,\Delta\left(\frac{d^2\varphi_k}{dx^2}\right) = -\ M\rho^2\,p_k^2\,\frac{d\varphi_k}{dx}(\ell_1) \tag{10b}$$

The solution of Equation (9) within either the region $0 \le x \le \ell_1$ or $\ell_1 \le x \le \ell$ must be of the form

$$\varphi_k = A_k \sin\beta_k x + B_k \cos\beta_k x + C_k \sinh\beta_k x + D_k \cosh\beta_k x \tag{11}$$

where $$\beta_k^4 = \frac{\mu p_k^2}{EI} \tag{12}$$

Since separate expressions for φ_k must be written for $0 \le x \le \ell_1$ and for $\ell_1 \le x \le \ell$, there are nine constants in the expression for each function

$\varphi_k(x)$ over the entire length of the beam.

As mentioned above, each function $\varphi_k(x)$ must also satisfy all boundary conditions. These conditions will consist of two boundary conditions from each end of the beam, plus the requirements that the deflection and the slope be continuous at $x = \ell_1$, and that the discontinuities in the moment and shear be described by Equations (10). Substitution of Equation (11) into these eight conditions gives an eigenvalue equation (frequency equation) from which one can determine the infinite sequence of discrete values of β_k, in terms of the physical parameters of the system, for which a function φ_k exists. One also obtains, by this substitution, relationships among the other eight constants in the expression for φ_k, such that only one constant remains arbitrary.

Thus a discrete set of functions $\varphi_k(x)$ which each satisfy all conditions for free vibration of the system does exist. Each of these functions represents a mode shape which will vibrate at an arbitrary amplitude in free vibration, with its own characteristic circular frequency. The circular frequencies, p_k, are determined from Equation (12) after finding the sequence of β_k values from the eigenvalue equation. Hence the functions φ_k are the normal mode shapes of the system and the frequencies p_k are the natural frequencies of the system.

Orthogonality Relationship. Similar to Equations (9) and (10) one can write

$$\frac{d^4\varphi_j}{dx^4} - \frac{\mu}{EI} p_j^2 \varphi_j = 0 \qquad (x \neq \ell_1) \qquad (13)$$

etc., for the j^{th} mode. Multiplying Equation (9) by $\varphi_j(x)$ and Equation (13) by $\varphi_k(x)$, subtracting one equation from the other, then integrating over the length of the beam gives

$$\left(p_k^2 - p_j^2\right) \frac{\mu}{EI} \int_0^{\ell} \varphi_k \varphi_j \, dx = \int_0^{\ell_1} \left(\frac{d^4\varphi_k}{dx^4} \varphi_j - \varphi_k \frac{d^4\varphi_j}{dx^4} \right) dx$$

$$+ \int_{\ell_1}^{\ell} \left(\frac{d^4\varphi_k}{dx^4} \varphi_j - \varphi_k \frac{d^4\varphi_j}{dx^4} \right) dx \, . \tag{14}$$

The left hand side of the equation can be written as one integral since the displacements are continuous at $x = \ell_1$. By performing four successive integrations by parts one can show that

$$\int_a^b \frac{d^4\varphi_k}{dx^4} \varphi_j \, dx = \left(\frac{d^3\varphi_k}{dx^3} \varphi_j - \frac{d^2\varphi_k}{dx^2} \frac{d\varphi_j}{dx} + \frac{d\varphi_k}{dx} \frac{d^2\varphi_j}{dx^2} - \varphi_k \frac{d^3\varphi_j}{dx^3} \right) \Bigg|_a^b + \int_a^b \varphi_k \frac{d^4\varphi_j}{dx^4} \, dx \tag{15}$$

Thus Equation (14) becomes

$$\left(p_k^2 - p_j^2\right) \frac{\mu}{EI} \int_0^{\ell} \varphi_k \varphi_j \, dx = \left(\frac{d^3\varphi_k}{dx^3} \varphi_j - \frac{d^2\varphi_k}{dx^2} \frac{d\varphi_j}{dx} + \frac{d\varphi_k}{dx} \frac{d^2\varphi_j}{dx^2} - \varphi_k \frac{d^3\varphi_j}{dx^3} \right) \Bigg|_0^{\ell_1}$$

$$+ \left(\frac{d^3\varphi_k}{dx^3} \varphi_j - \frac{d^2\varphi_k}{dx^2} \frac{d\varphi_j}{dx} + \frac{d\varphi_k}{dx} \frac{d^2\varphi_j}{dx^2} - \varphi_k \frac{d^3\varphi_j}{dx^3} \right) \Bigg|_{\ell_1}^{\ell} \tag{16}$$

Each of the terms on the right hand side of Equation (16) is zero at the ends of the beam for the boundary conditions most frequently considered; namely fixed, free, hinged or sliding. This is obviously true since either the shear or the deflection is zero at the boundary in every case, and also either the bending moment or the slope is zero at the boundary. Thus for beams with these particular end conditions the only contribution of the right hand side of Equation (16) comes from the discontinuities of the second derivative and third derivative terms at $x = \ell_1$. Hence Equation (16) becomes

$$(p_k^2 - p_j^2)\,\frac{\mu}{EI}\int_0^{\ell} \varphi_k \varphi_j \, dx = \varphi_k(\ell_1)\,\Delta\left(\frac{d^3\varphi_j}{dx^3}\right) - \frac{d\varphi_k}{dx}(\ell_1)\,\Delta\left(\frac{d^2\varphi_j}{dx^2}\right)$$

$$- \varphi_j(\ell_1)\,\Delta\left(\frac{d^3\varphi_k}{dx^3}\right) + \frac{d\varphi_j}{dx}(\ell_1)\,\Delta\left(\frac{d^2\varphi_k}{dx^2}\right) \quad . \qquad (17)$$

Substituting Equations (10) and similar equations for φ_j into (17) gives

$$\frac{1}{EI}\,(p_k^2 - p_j^2)\left(\mu\int_0^{\ell} \varphi_k \varphi_j \, dx + M\varphi_k(\ell_1)\varphi_j(\ell_1) + M\rho^2\,\frac{d\varphi_k}{dx}(\ell_1)\,\frac{d\varphi_j}{dx}(\ell_1)\right) = 0$$

When $k \neq j$, the frequency term $(p_k^2 - p_j^2)$ does not equal zero. Thus, the orthogonality relationship for the normal modes of a slender beam carrying one discrete load having both mass and rotational inertia can be written as

$$\mu \int_0^{\ell} \varphi_k \varphi_j dx + M\varphi_k(\ell_1)\varphi_j(\ell_1) + M\rho^2 \frac{d\varphi_k}{dx}(\ell_1) \frac{d\varphi_j}{dx}(\ell_1) = 0$$

$$(k \neq j) \qquad (18)$$

Response to Support Excitation. If the system under consideration is excited by motion of the supports of the beam, the equations for the relative motion are:

$$EI \frac{\partial^4 y}{\partial x^4} + \mu \frac{\partial^2 y}{\partial t^2} = -\mu \ddot{s} \qquad (x \neq \ell_1) \qquad (19)$$

$$EI\, \Delta\left(\frac{\partial^3 y}{\partial x^3}\right) + M \frac{\partial^2 y}{\partial t^2}(\ell_1) = -M\ddot{s} \qquad (20)$$

$$EI\, \Delta\left(\frac{\partial^2 y}{\partial x^2}\right) - M\rho^2 \frac{\partial^3 y}{\partial x \partial t^2}(\ell_1) = 0 \qquad (21)$$

As mentioned above the $\varphi_k(x)$ functions form a complete set of orthonormal eigenfunctions and, hence, one can express the displacement as

$$y(x,t) = \sum_{k=1}^{\infty} \xi_k(t)\, \varphi_k(x) \quad . \qquad (22)$$

Substituting this expression into Equation (19) gives

$$EI \sum_{k=1} \xi_k \frac{d^4\varphi_k}{dx^4} + \mu \sum_{k=1}^{\infty} \ddot{\xi}_k \varphi_k = -\mu \ddot{s} \quad . \qquad (23)$$

Using the relationship

$$\frac{d^4\varphi_k}{dx^4} = \frac{\mu p_k^2}{EI} \varphi_k(x)$$

from Equation (8) gives

$$\sum_{k=1}^{\infty} \left(\mu p_k^2 \xi_k + u \ddot{\xi}_k\right) \varphi_k = -\mu \ddot{s} \quad . \tag{24}$$

Multiplying both sides of Equation (24) by $\varphi_j(x)$ and integrating over the length of the beam gives

$$\sum_{k=1}^{\infty} \left(p_k^2 \xi_k + \ddot{\xi}_k\right) \mu \int_0^{\ell} \varphi_k \varphi_j \, dx = -\mu \ddot{s} \int_0^{\ell} \varphi_j \, dx \quad . \tag{25}$$

Similarly substituting Equation (22) into Equations (20) and (21) and using Equations (10) to simplify gives

$$\sum_{k=1}^{\infty} \left(p_k^2 \xi_k + \ddot{\xi}_k\right) M \varphi_k(\ell_1) = - M \ddot{s} \tag{26}$$

$$\sum_{k=1}^{\infty} \left(p_k^2 \xi_k + \ddot{\xi}_k\right) M \rho^2 \frac{d\varphi_k}{dx}(\ell_1) = 0 \quad . \tag{27}$$

Multiplying both sides of Equation (26) by $\varphi_j(\ell_1)$ and both sides of Equation (27) by $\frac{d\varphi_j}{dx}(\ell_1)$ then adding the two resulting equations to Equation (25) gives

$$\sum_{k=1}^{\infty} \left(p_k^2 \xi_k + \ddot{\xi}_k\right) \left(\mu \int_0^{\ell} \varphi_k \varphi_j dx + M \varphi_k(\ell_1) \varphi_j(\ell_1) + M \rho^2 \frac{d\varphi_k}{dx}(\ell_1) \frac{d\varphi_j}{dx}(\ell_1) \right) = -\ddot{s} \left(\mu \int_0^{\ell} \varphi_j dx + M \varphi_j(\ell_1) \right). \tag{28}$$

From the orthogonality relationship, Equation (18), it is apparent that every term in the infinite series in Equation (28) is identically zero except the term for k = j. Thus

$$(p_k^2 \xi_k + \ddot{\xi}_k)\left(\mu \int_0^{\ell} \varphi_k^2\, dx + M\varphi_k^2(\ell_1)\right.$$

$$\left. + M\rho^2 \left[\frac{d\varphi_k}{dx}(\ell_1)\right]^2\right) = -\ddot{s}\left(\mu \int_0^{\ell} \varphi_k dx + M\varphi_k(\ell_1)\right) \tag{29}$$

or

$$\ddot{\xi}_k + p_k^2 \xi_k = -\eta_k \ddot{s} \tag{30}$$

where

$$\eta_k = \frac{\mu \int_0^{\ell} \varphi_k\, dx + M\varphi_k(\ell_1)}{\mu \int_0^{\ell} \varphi_k^2 dx + M\varphi_k^2(\ell_1) + M\rho^2\left(\frac{d\varphi_k}{dx}(\ell_1)\right)^2} \tag{31}$$

Equation (30) is identical in form to Equation (1), the equation for a single degree of freedom system. Thus for any type of support motion for which the response of the single degree of freedom oscillator is known, one can easily calculate the response of each mode of the beam carrying one discrete load. The factor η_k is usually called the mode participation factor since it determines the amount of excitation of each separate mode for a given excitation of the entire system.

Numerical Results. The frequency equations of the beams under consideration here can be written in the form

$$\frac{M}{m}\, \alpha_k^3 \left(\frac{\rho}{\ell}\right)^2 = \frac{K_a + \frac{M}{m}\, \alpha_k K_b}{K_c + \frac{M}{m}\, \alpha_k K_d} \tag{32}$$

where $m = \mu\ell$, the mass of the beam, $\alpha_k = \beta_k \ell$, and the K are defined in

Table 1 for three particular sets of beam end conditions. The numerical values of the first five roots ($\alpha_k = \beta_k \ell$) of the frequency equations for the beam configurations illustrated in Figure 2 are presented graphically. The natural frequencies of vibration are related to these roots by Equation (12). References 8, 9, and 10 also present numerical values for the natural frequencies of some of the systems considered here and for some related systems.

Expressions for the characteristic mode shapes of the beams shown in Figure 2 are given in Table 2. The numerical values of the coefficients in these expressions for the first five mode shapes are presented graphically, and tabulated values of the combined trigonometric-hyperbolic functions involved are available in References 1, 2, and 11. The amplitudes of the mode shapes were normalized by assigning a particular value to the generalized mass,

$$\mathcal{M}_k = \mu \int_0^{\ell} \varphi_k^2 dx + M\varphi_k^2(\ell_1) + M\rho^2 \left(\frac{d\varphi_k}{dx}(\ell_1) \right)^2 . \tag{33}$$

For the fixed-free and fixed-fixed beams the value of $\mathcal{M}_k$ was set equal to m in each case, and for the hinged-hinged beams $\mathcal{M}_k$ was set equal to m/2. This agrees with common practice for normalizing the mode shapes of uniform beams without discrete loads.

Numerical values of the mode participation factors, as defined by Equation (31), for the first five modes of the beams considered here are also presented graphically. Graphs are not presented for the even-numbered modes of the hinged-hinged and fixed-fixed beams with the discrete load at mid-span since the mode participation factor is identically zero for such anti-symmetric cases.

For comparison Table 3 presents the numerical values of the various mode factors for the limiting situation of a beam without a discrete load.

The following curves giving numerical values of beam characteristics have been arranged according to the following scheme:

Beam Type	Mode Number	Response Characteristic
A. Fixed-Free	1	a. Frequency Roots
B. Hinged-Hinged (Fig. 2)	2	b. Mode Shape Factors
C. " "	3	c. Mode Participation Factors
D. " "	4	
E. Fixed-Fixed (Fig. 2)	5	
F. " "		
G. " "		

Thus, the curve labeled C-3-c gives the mode participation factor for the third mode of a hinged-hinged beam with the load at the one-third point.

The curves are grouped together according to first beam type, next mode number, and then response characteristic.

For the fixed-fixed beam with the discrete load at the one-sixth point there are two particular combinations of parameters which give a rather ill-conditioned problem. When $M/m \approx 3.01$ and $\rho/\ell \approx 0.13$ the second and third modes have nearly identical natural frequencies, and when $M/m \approx 0.31$ and $\rho/\ell \approx 0.13$ the third and fourth modes have nearly identical natural frequencies. Curves G-2, 3, 4 show the resulting discontinuous and nearly discontinuous mode factors. From curves D-2, 3 it appears that a similar situation may exist for the second and third modes of the hinged-hinged beam with the discrete load at the one-sixth point for some value of M/m between 1.0 and 3.0.

The use of the curves can best be explained by giving a specific numerical example.

Example: Suppose that the beam has fixed-fixed supports and a load at the one-third point.

The discrete mass is equal to the mass of the beam. The radius of gyration of the discrete mass is one-tenth of the length of the beam.

1.) Frequencies of Vibration. From curves F-1, 2, 3, 4, 5-a we find, corresponding to $(\rho/\ell) = 0.1$ and $(M/m) = 1$, $\alpha_1 = 3.63$, $\alpha_2 = 6.15$, $\alpha_3 = 7.78$, $\alpha_4 = 11.98$, and $\alpha_5 = 14.47$. Referring to Equation (12), and noting that $\alpha_k = \beta_k \ell$, we find:

$$p_1 = \frac{13.2}{\ell^2}\sqrt{\frac{EI}{\mu}} \; ; \; p_2 = \frac{37.8}{\ell^2}\sqrt{\frac{EI}{\mu}} \; ; \; p_3 = \frac{60.5}{\ell^2}\sqrt{\frac{EI}{\mu}} \; ;$$

$$p_4 = \frac{143}{\ell^2}\sqrt{\frac{EI}{\mu}} \; ; \; p_5 = \frac{210}{\ell^2}\sqrt{\frac{EI}{\mu}} \; .$$

It is of interest to compare these values with those for a negligible radius of gyration of the mass, which gives

$$p_1 = \frac{13.6}{\ell^2}\sqrt{\frac{EI}{\mu}} \; ; \; p_2 = \frac{46.9}{\ell^2}\sqrt{\frac{EI}{\mu}} \; ; \; p_3 = \frac{119}{\ell^2}\sqrt{\frac{EI}{\mu}} \; ;$$

$$p_4 = \frac{178}{\ell^2}\sqrt{\frac{EI}{\mu}} \; ; \; p_5 = \frac{262}{\ell^2}\sqrt{\frac{EI}{\mu}} \; .$$

2.) Mode Shapes. To find the mode shapes first look up on curves F-1, 2, 3, 4, 5-b the numerical factors A_k, B_k, C_k, and D_k. These factors are then put into the equations in Table 2 for the fixed-fixed beam, from which the mode shapes are:

$$\varphi_1(x_1) = 1.12(\cosh 3.63 \frac{x_1}{\ell} - \cos 3.63 \frac{x_1}{\ell}) - 1.46(\sinh 3.63 \frac{x_1}{\ell} - \sin 3.63 \frac{x_1}{\ell})$$

$$\varphi_1(x_2) = 0.763(\cosh 3.63 \frac{x_2}{\ell} - \cos 3.63 \frac{x_2}{\ell}) - 0.835(\sinh 3.63 \frac{x_2}{\ell} - \sin 3.63 \frac{x_2}{\ell})$$

$$\varphi_2(x_1) = 0.910(\cosh 6.15 \frac{x_1}{\ell} - \cos 6.15 \frac{x_1}{\ell}) - 1.24(\sinh 6.15 \frac{x_1}{\ell} - \sin 6.15 \frac{x_1}{\ell})$$

$$\varphi_2(x_2) = -0.665(\cosh 6.15 \frac{x_2}{\ell} - \cos 6.15 \frac{x_2}{\ell}) + 0.673(\sinh 6.15 \frac{x_2}{\ell} - \sin 6.15 \frac{x_2}{\ell})$$

$$\varphi_3(x_1) = 0.25(\cosh 7.78 \frac{x_1}{\ell} - \cos 7.78 \frac{x_1}{\ell}) - 0.34(\sinh 7.78 \frac{x_1}{\ell} - \sin 7.78 \frac{x_1}{\ell})$$

$$\varphi_3(x_2) = 1.035(\cosh 7.78 \frac{x_2}{\ell} - \cos 7.78 \frac{x_2}{\ell}) - 1.021(\sinh 7.78 \frac{x_2}{\ell} - \sin 7.78 \frac{x_2}{\ell})$$

$$\varphi_4(x_1) = -0.00305\, e^{11.98\, x_1/\ell}$$

$$\varphi_4(x_2) = (4.02 \times 10^{-4})\, e^{11.98\, x_2/\ell} - 1.20\,(e^{-11.98\, x_2/\ell} - \cos 11.98 \frac{x_2}{\ell} + \sin 11.98 \frac{x_2}{\ell})$$

$$\varphi_5(x_1) = 0.0132\, e^{14.47\, x_1/\ell} + 1.68\, e^{-14.47\, x_1/\ell} - 1.70 \cos 14.47 \frac{x_1}{\ell} + 1.67 \sin 14.47 \frac{x_1}{\ell}$$

$$\varphi_5(x_2) = -(1.1 \times 10^{-5})e^{14.47\, x_2/\ell} - 0.04\left(e^{-14.47\, x_2/\ell} - \cos 14.47 \frac{x_2}{\ell} + \sin 14.47 \frac{x_2}{\ell}\right).$$

3.) Mode Participation Factors. The mode participation factors η_k are found from curves F-1, 2, 3, 4, 5-c, which yield:

$$\eta_1 = 1.26 \ , \quad \eta_2 = 0.19 \ , \quad \eta_3 = 0.35 \ , \quad \eta_4 = 0.20 \ ,$$

$$\eta_5 = 0.23 \ .$$

These factors, when put into Equation (30), can be used for total response calculations.

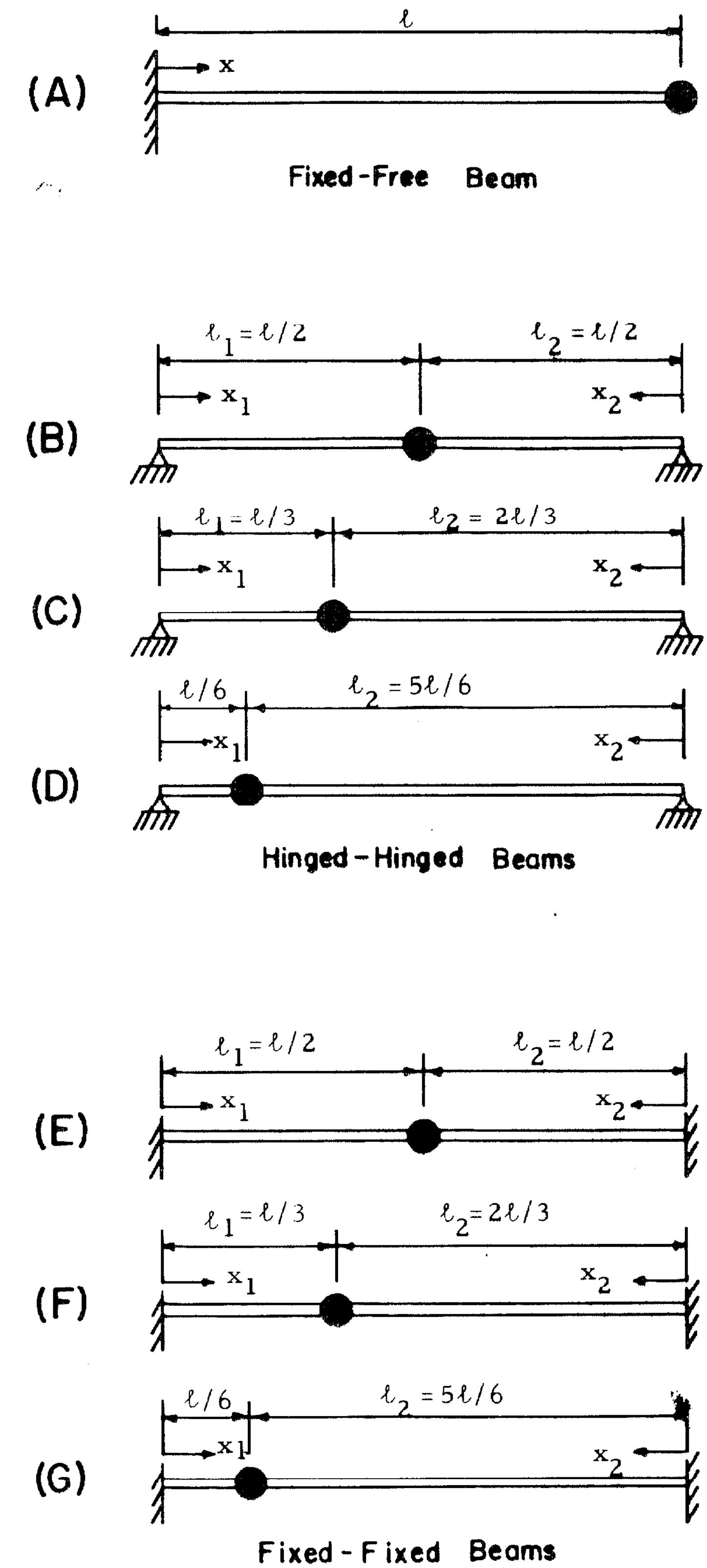


Figure 2. Beams for Which Numerical Results are Presented.

Table 1

Terms in the Frequency Equation (32)

Fixed-Free Beam:

$$K_a = \cos\beta_k \ell \cosh\beta_k \ell + 1$$

$$K_b = \cos\beta_k \ell \sinh\beta_k \ell - \sin\beta_k \ell \cosh\beta_k \ell$$

$$K_c = \cos\beta_k \ell \sinh\beta_k \ell + \sin\beta_k \ell \cosh\beta_k \ell$$

$$K_d = \cos\beta_k \ell \cosh\beta_k \ell - 1$$

Hinged-Hinged Beam:

$$K_a = -2 \sin\beta_k \ell \sinh\beta_k \ell$$

$$K_b = \sin\beta_k \ell_1 \sin\beta_k \ell_2 \sinh\beta_k \ell - \sinh\beta_k \ell_1 \sinh\beta_k \ell_2 \sin\beta_k \ell$$

$$K_c = \cos\beta_k \ell_1 \cos\beta_k \ell_2 \sinh\beta_k \ell - \cosh\beta_k \ell_1 \cosh\beta_k \ell_2 \sin\beta_k \ell$$

$$K_d = (\cos\beta_k \ell_1 \sinh\beta_k \ell_1 - \sin\beta_k \ell_1 \cosh\beta_k \ell_1)(\cos\beta_k \ell_2 \sinh\beta_k \ell_2 - \sin\beta_k \ell_2 \cosh\beta_k \ell_2)$$

Fixed-Fixed Beam:

$$K_a = 2(\cos\beta_k \ell \cosh\beta_k \ell - 1)$$

$$K_b = (\cos\beta_k \ell_1 \cosh\beta_k \ell_1 - 1)(\cos\beta_k \ell_2 \sinh\beta_k \ell_2 - \sin\beta_k \ell_2 \cosh\beta_k \ell_2) + (\cos\beta_k \ell_2 \cosh\beta_k \ell_2 - 1)(\cos\beta_k \ell_1 \sinh\beta_k \ell_1 - \sin\beta_k \ell_1 \cosh\beta_k \ell_1)$$

$$K_c = (\cos\beta_k \ell_1 \cosh\beta_k \ell_1 - 1)(\cos\beta_k \ell_2 \sinh\beta_k \ell_2 + \sin\beta_k \ell_2 \cosh\beta_k \ell_2) + (\cos\beta_k \ell_2 \cosh\beta_k \ell_2 - 1)(\cos\beta_k \ell_1 \sinh\beta_k \ell_1 + \sin\beta_k \ell_1 \cosh\beta_k \ell_1)$$

$$K_d = (\cos\beta_k \ell_1 \cosh\beta_k \ell_1 - 1)\cos\beta_k \ell_2 \cosh\beta_k \ell_2 - 1)$$

Table 2

Characteristic Mode Shapes for Uniform Beams
with One Discrete Load

Fixed-Free Beam

$$\varphi_k(x) = B_k(\cosh\beta_k x - \cos\beta_k x) + D_k(\sinh\beta_k x - \sin\beta_k x)$$

or

$$\varphi_k(x) = \frac{B_k+D_k}{2} e^{\beta_k x} + \frac{B_k-D_k}{2} e^{-\beta_k x} - B_k\cos\beta_k x - D_k\sin\beta_k x$$

Hinged-Hinged Beam

$$\varphi_k(x_1) = A_k \sin\beta_k x_1 + B_k \sinh\beta_k x_1 \qquad (0 \le x_1 \le \ell_1)$$

$$\varphi_k(x_2) = C_k \sin\beta_k x_2 + D_k \sinh\beta_k x_2 \qquad (0 \le x_2 \le \ell_2)$$

Fixed-Fixed Beam

$$\varphi_k(x_1) = A_k(\cosh\beta_k x_1 - \cos\beta_k x_1) + B_k(\sinh\beta_k x_1 - \sin\beta_k x_1)$$

or $(0 \le x_1 \le \ell_1)$

$$\varphi_k(x_1) = \frac{A_k+B_k}{2} e^{\beta_k x_1} + \frac{A_k-B_k}{2} e^{-\beta_k x_1} - A_k\cos\beta_k x_1 - B_k\sin\beta_k x_1$$

$$\varphi_k(x_2) = C_k(\cosh\beta_k x_2 - \cos\beta_k x_2) + D_k(\sinh\beta_k x_2 - \sin\beta_k x_2)$$

or $(0 \le x_2 \le \ell_2$

$$\varphi_k(x_2) = \frac{C_k+D_k}{2} e^{\beta_k x_2} + \frac{C_k-D_k}{2} e^{-\beta_k x_2} - C_k\cos\beta_k x_2 - D_k\sin\beta_k x_2$$

Table 3

Mode Factors for Uniform Beams
Without Discrete Loads

Fixed-Free

Factor	First Mode	Second Mode	Third Mode	Fourth Mode	Fifth Mode
α_k	1.8751	4.6941	7.8548	10.9955	14.1372
B_k	1.0000	1.0000	1.0000	1.0000	1.0000
$B_k + D_k$	0.2659	-0.0185	7.75×10^{-4}	-3.36×10^{-5}	1.46×10^{-6}
η_k	0.7830	0.4339	0.2544	0.1819	0.1415

Hinged-Hinged

Factor	First Mode	Second Mode	Third Mode	Fourth Mode	Fifth Mode
α_k	3.1416	6.2832	9.4248	12.5664	15.7080
A_k	1.0000	-1.0000	1.0000	1.0000	1.0000
B_k	0.0000	0.0000	0.0000	0.0000	0.0000
C_k	1.0000	1.0000	1.0000	-1.0000	1.0000
D_k	0.0000	0.0000	0.0000	0.0000	0.0000
η_k	1.2732	0.0000	0.4244	0.0000	0.2546

Fixed-Fixed

Factor	First Mode	Second Mode	Third Mode	Fourth Mode	Fifth Mode
α_k	4.7300	7.8532	10.9956	14.1372	17.2788
A_k	1.0000	1.0000	1.0000	1.0000	1.0000
$A_k + B_k$	0.0175	-7.77×10^{-4}	3.35×10^{-5}	-1.46×10^{-6}	6.26×10^{-8}
C_k	1.0000	-1.0000	1.0000	-1.0000	1.0000
$C_k + D_k$	0.0175	7.77×10^{-4}	3.35×10^{-5}	1.46×10^{-6}	6.26×10^{-8}
η_k	0.8386	0.0000	0.3638	0.0000	0.2316

Summary of Nomenclature

Symbol	Explanation or Definition
A_k, B_k, C_k, D_k	mode shape factors (Table 2)
EI	beam flexural rigidity
K_i	frequency equation factor (Eqn. 32)
M	discrete mass
$\mathcal{M}_k$	generalized mass (Eqn. 33)
j, k	subscripts denoting mode number
ℓ	beam length
ℓ_1, ℓ_2	location of discrete load on beam ($\ell_1 + \ell_2 = \ell$)
m	mass of beam ($m = \mu\ell$)
p_k	natural circular frequency
s	motion of support
$\ddot{s}$	acceleration of support
t	time
x	longitudinal coordinate
y	motion of beam relative to support
α_k	root of frequency equation ($\alpha_k = \beta_k \ell$)
β_k	frequency factor (Eqn. 12)
$\Delta\left(\frac{\partial^n y}{\partial x^n}\right)$	finite increase in $\frac{\partial^n y}{\partial x^n}$ at $x = \ell_1$
η_k	mode participation factor
μ	beam mass per unit length
ξ_k	time dependent coefficient of φ_k (Eqn. 3)
ρ	radius of gyration of discrete load about neutral plane of beam
φ_k	normal mode shape
ψ_k	time dependent coefficient of φ_k in free vibration

References

1. Vibration Analysis Tables, by R. E. D. Bishop and D. C. Johnson, Cambridge University Press, 1956 (extracted from Ref. 2).

2. The Mechanics of Vibration, by R. E. D. Bishop and D. C. Johnson, Cambridge University Press, 1960.

3. Tables of Characteristic Functions Representing Normal Modes of Vibration of a Beam, by Dana Young and R. P. Felgar, Engineering Research Bulletin No. 4913, Bureau of Engineering Research, University of Texas, 1949.

4. "Forced Lateral Vibration of Beam Carrying a Concentrated Mass," by W. H. Hoppmann, Transactions of the American Society of Mechanical Engineers, vol. 74, Journal of Applied Mechanics, vol. 19 (Sept. 1952), pp. 301-07.

5. "On the Vibration of Beams or Rods Carrying a Concentrated Mass," by Yu Chen, Transactions of the ASME, Series E, Journal of Applied Mechanics, vol. 30 (June 1963), pp. 310-11.

6. "Free Vibrations of Constrained Beams," by W. F. Z. Lees and Edward Saibel, ASME, vol. 74, Journal of Applied Mechanics, vol. 19 (Dec. 1952), pp. 471-77.

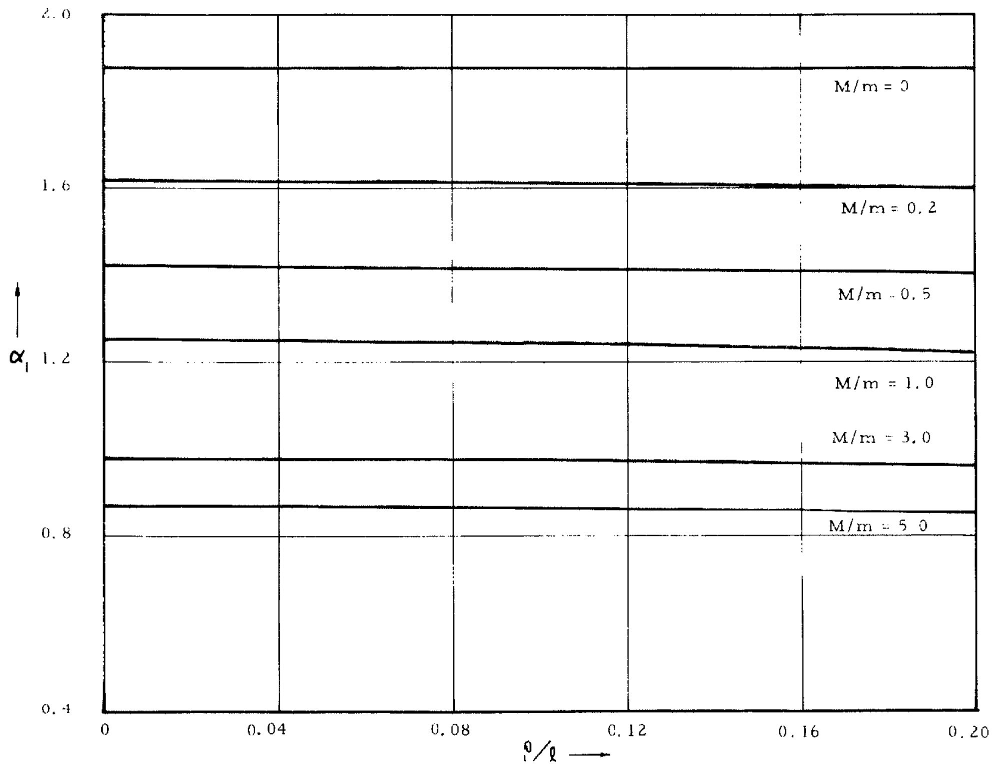
7. "Transverse Vibration of an Euler Beam Carrying a System of Heavy Bodies," by H. H. Pan, Transactions of the ASME, Series E, Journal of Applied Mechanics, vol. 32 (June 1965), pp. 434-37.

8. "The Influence of a Concentrated Mass on the Free Vibrations of a Uniform Beam," by J. C. Maltbaek, International Journal of Mechanical Science, Pergamon Press Ltd., 1961, vol. 3, pp. 197-218.

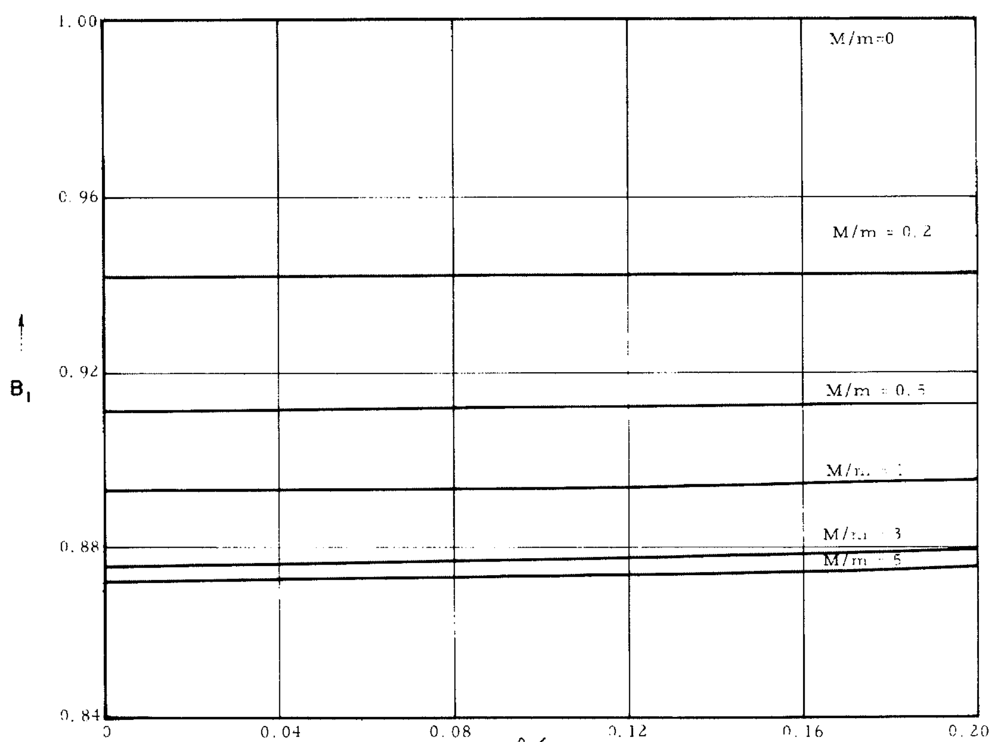
9. "Vibration Frequencies for Uniform Beams with Central Masses," by W. E. Baker, Transactions of the ASME, Series E., Journal of Applied Mechanics, vol. 31 (June 1964), pp. 335-37.

10. "Vibration Frequencies for a Uniform Beam with Central Mass and Elastic Supports," by Milton S. Hess, Transactions of the ASME, Series E, Journal of Applied Mechanics, vol. 31 (Sept. 1964), pp. 556-58.

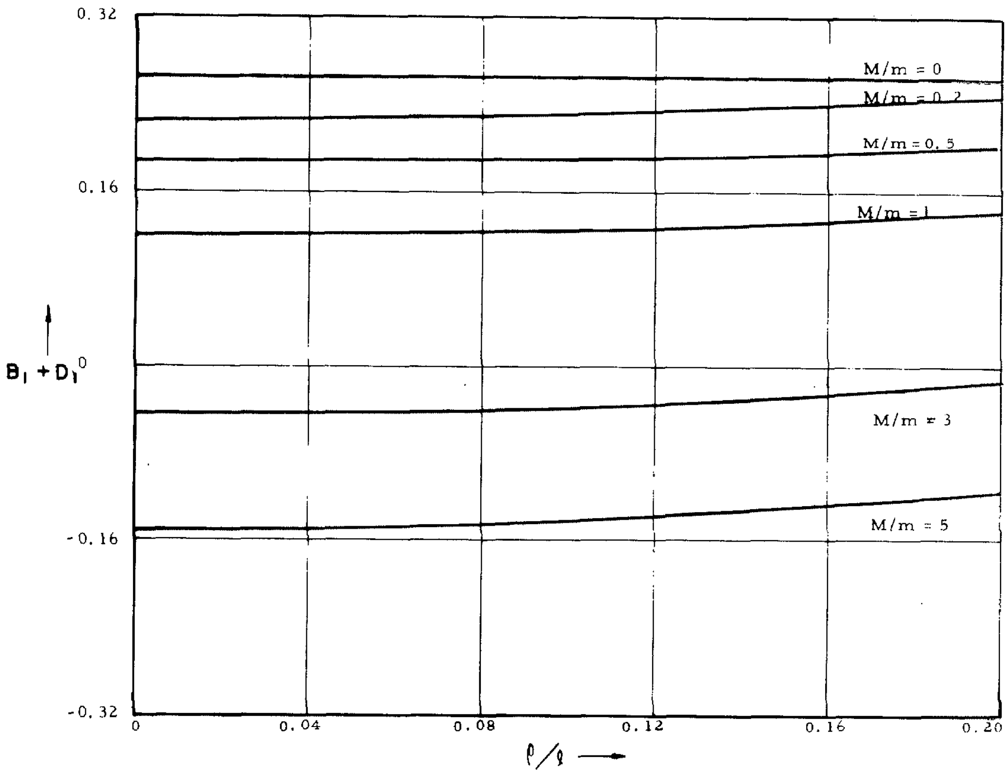
11. "The Analysis of Vibrating Systems which Embody Beams in Flexure," by R. E. D. Bishop, Proceedings of the Institution of Mechanical Engineers, vol. 169, no. 51 (1955), p. 1031-50.



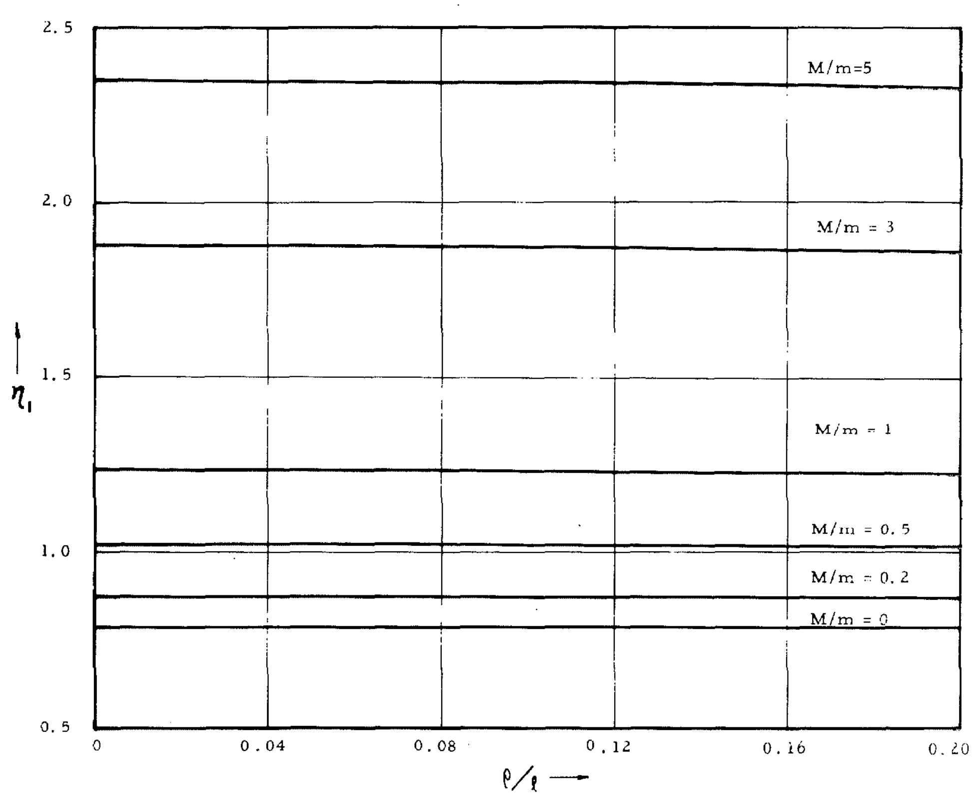
A-1-a FIXED-FREE - FIRST MODE - FREQUENCY ROOT



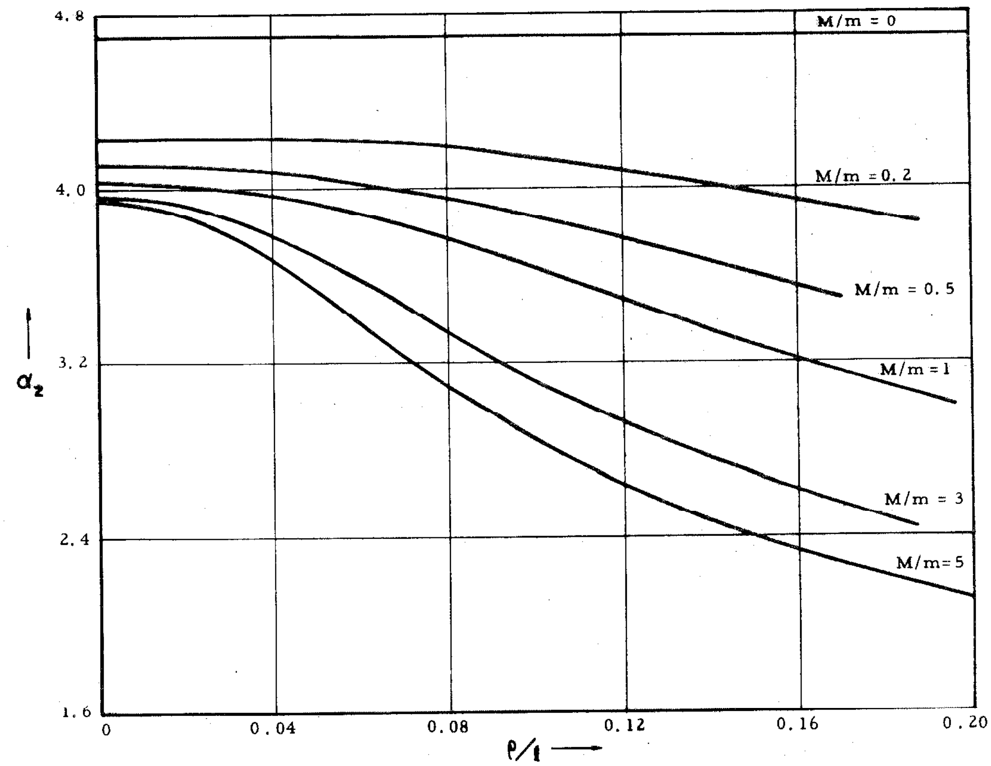
A-1-b(1). FIXED-FREE - FIRST MODE - MODE SHAPE FACTOR



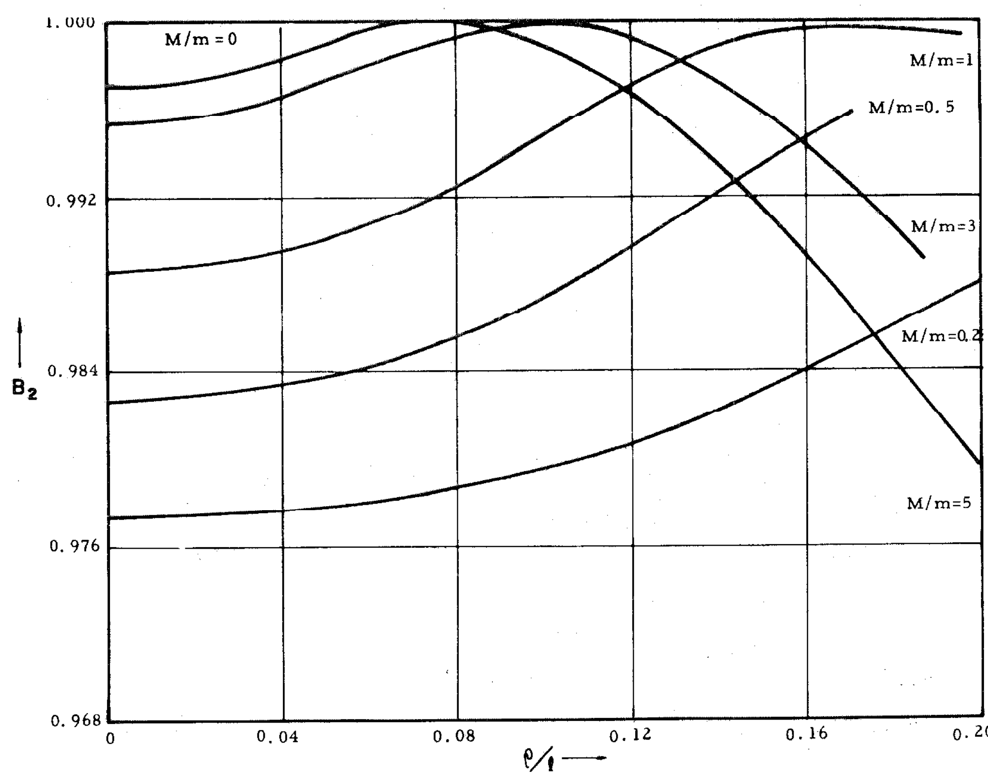
A-1-b(2). FIXED-FREE - FIRST MODE - MODE SHAPE FACTOR



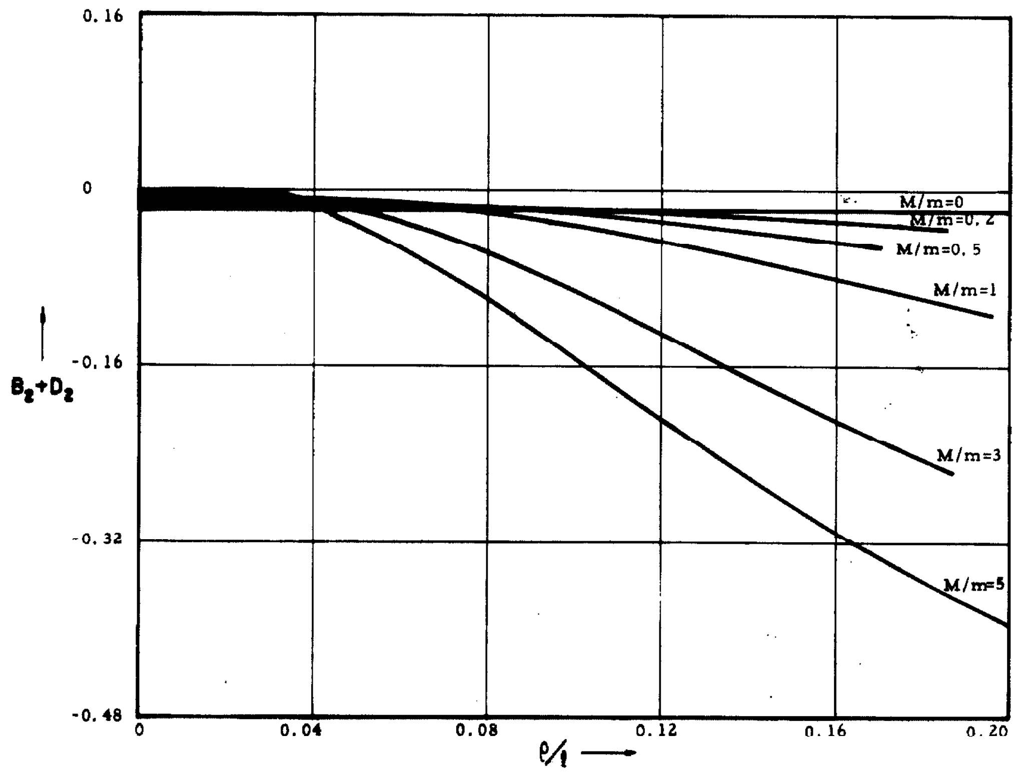
A-1-c FIXED-FREE - FIRST MODE - PARTICIPATION FACTOR



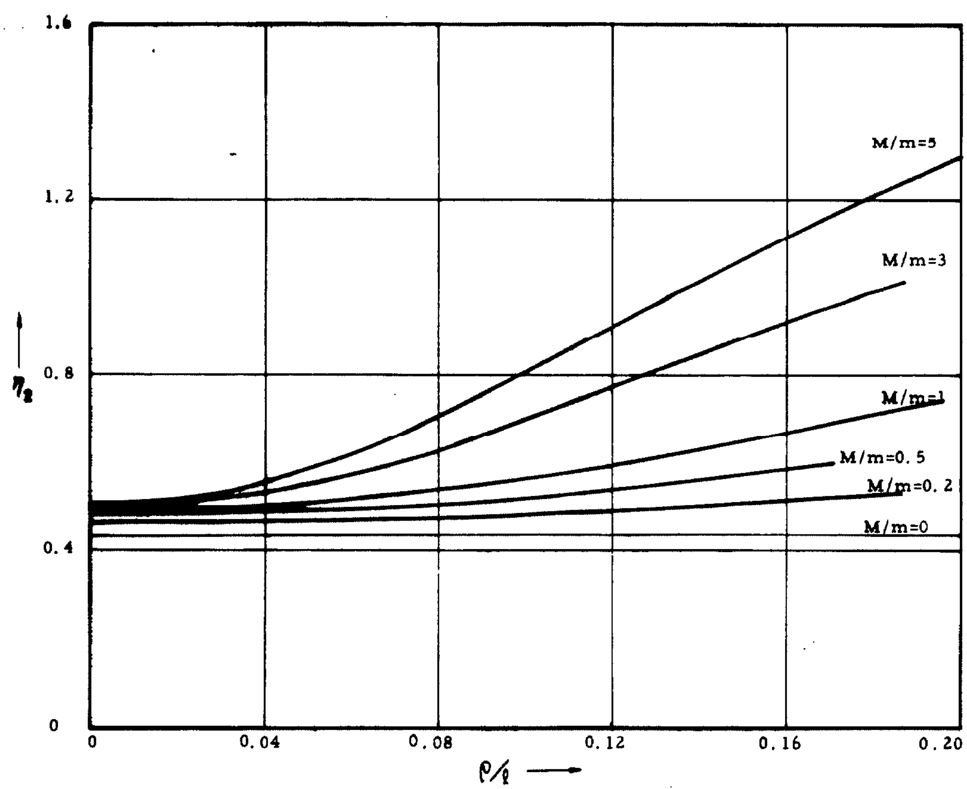
A-2-a FIXED-FREE-SECOND MODE - FREQUENCY ROOT



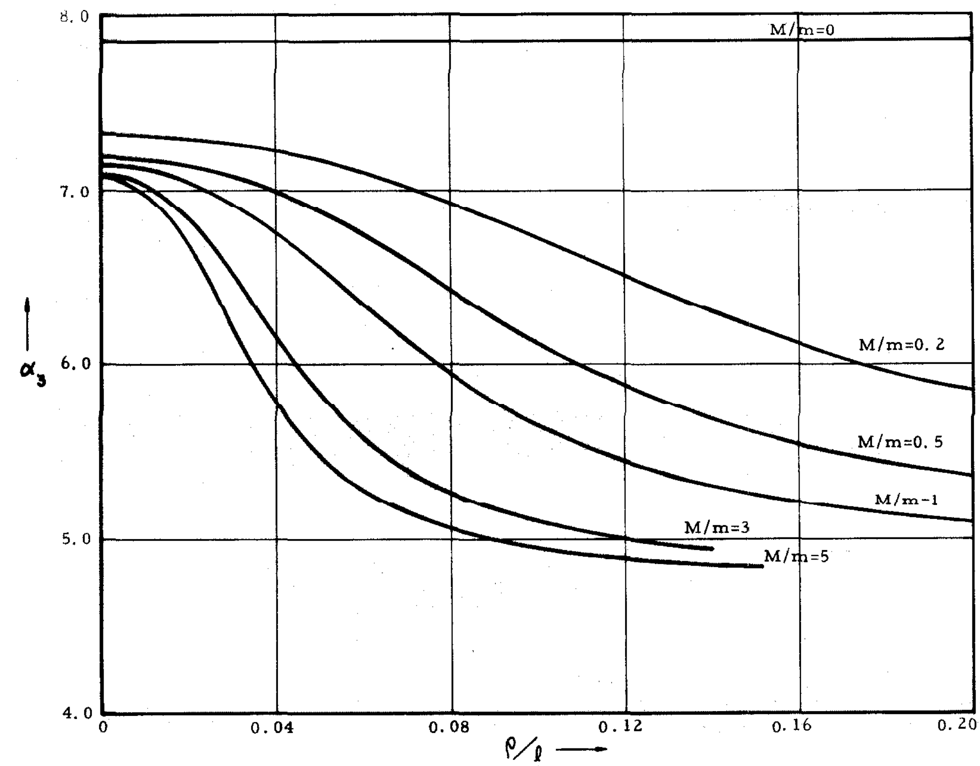
A-2-b(1). FIXED-FREE- SECOND MODE - MODE SHAPE FACTOR



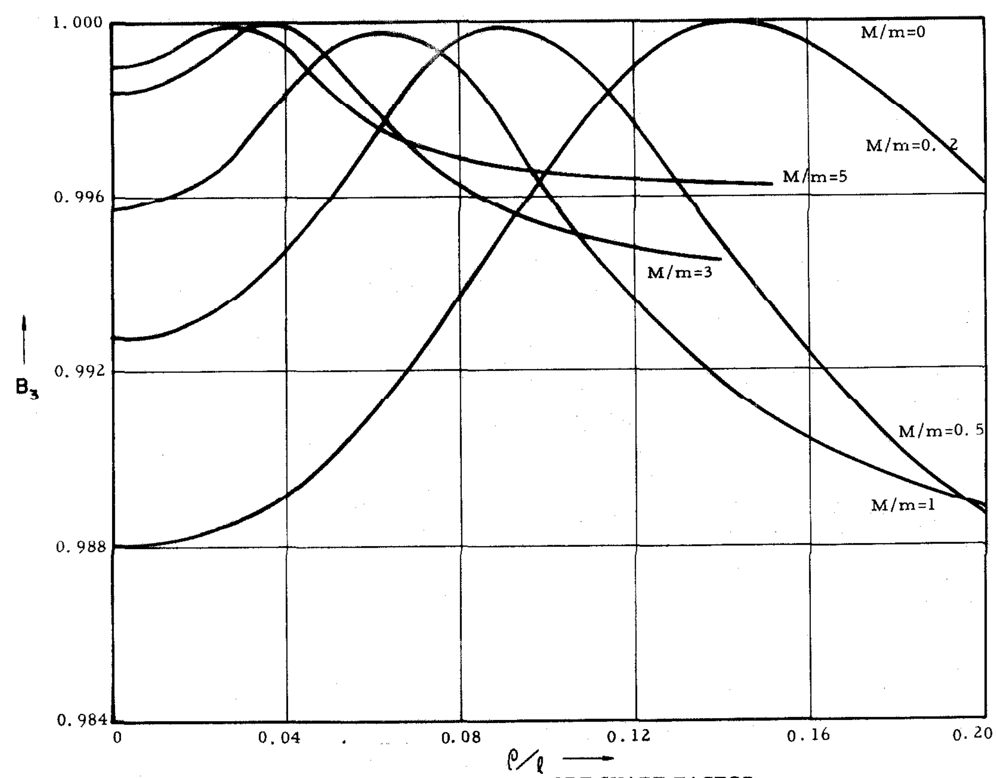
A-2-b(2). FIXED-FREE - SECOND MODE - MODE SHAPE FACTOR



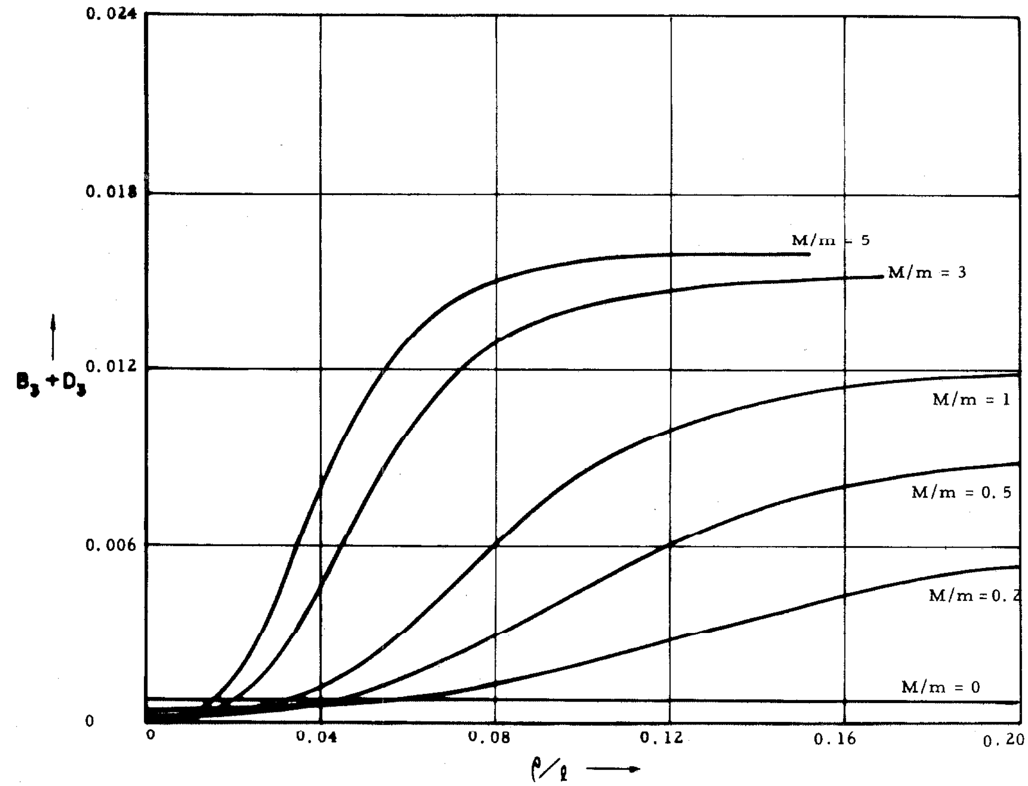
A-2-c. FIXED-FREE- SECOND MODE - MODE PARTICIPATION FACTOR



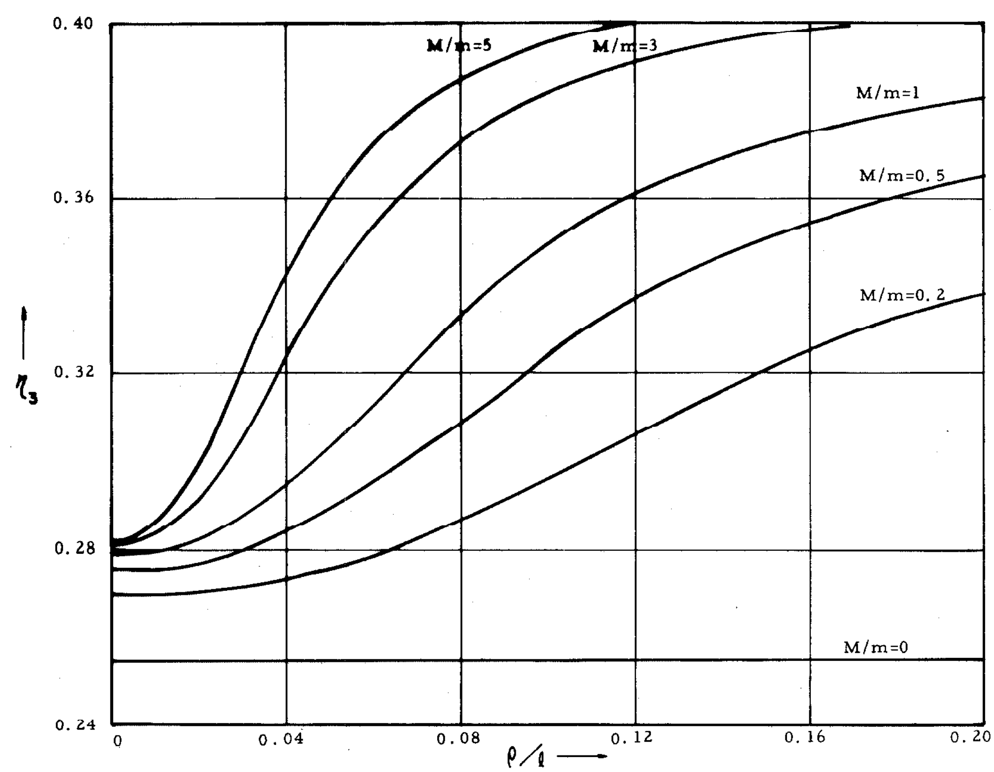
A-3-a. FIXED-FREE- THIRD MODE - FREQUENCY ROOT



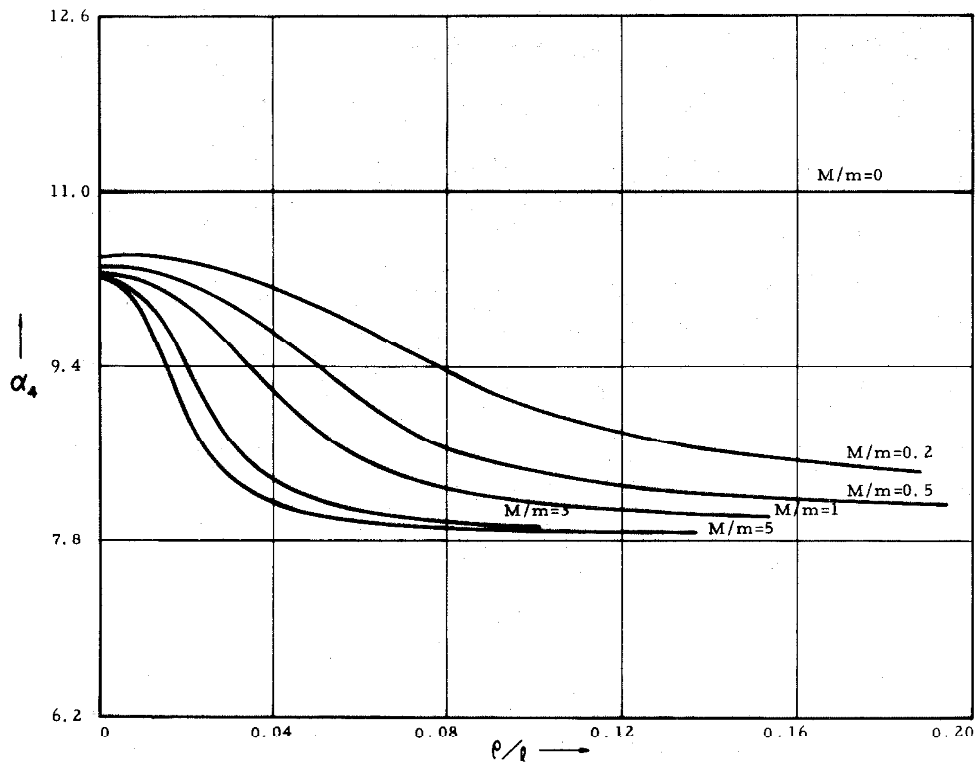
A-3-b(1). FIXED-FREE- THIRD MODE - MODE SHAPE FACTOR



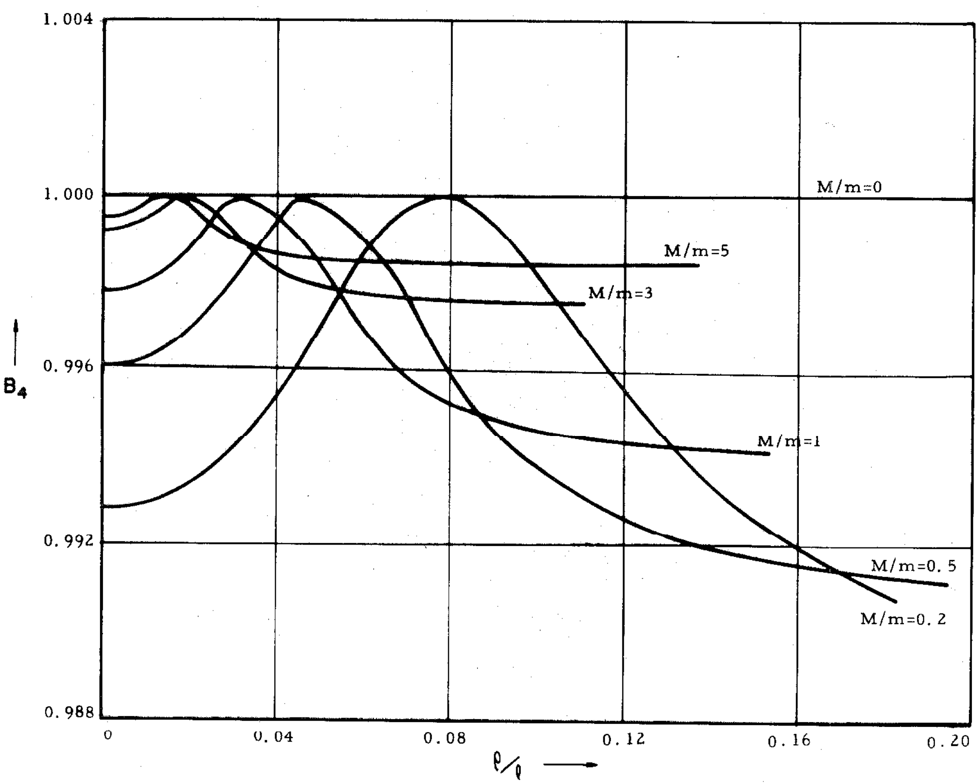
A-3-b(2). FIXED-FREE - THIRD MODE - MODE SHAPE FACTOR



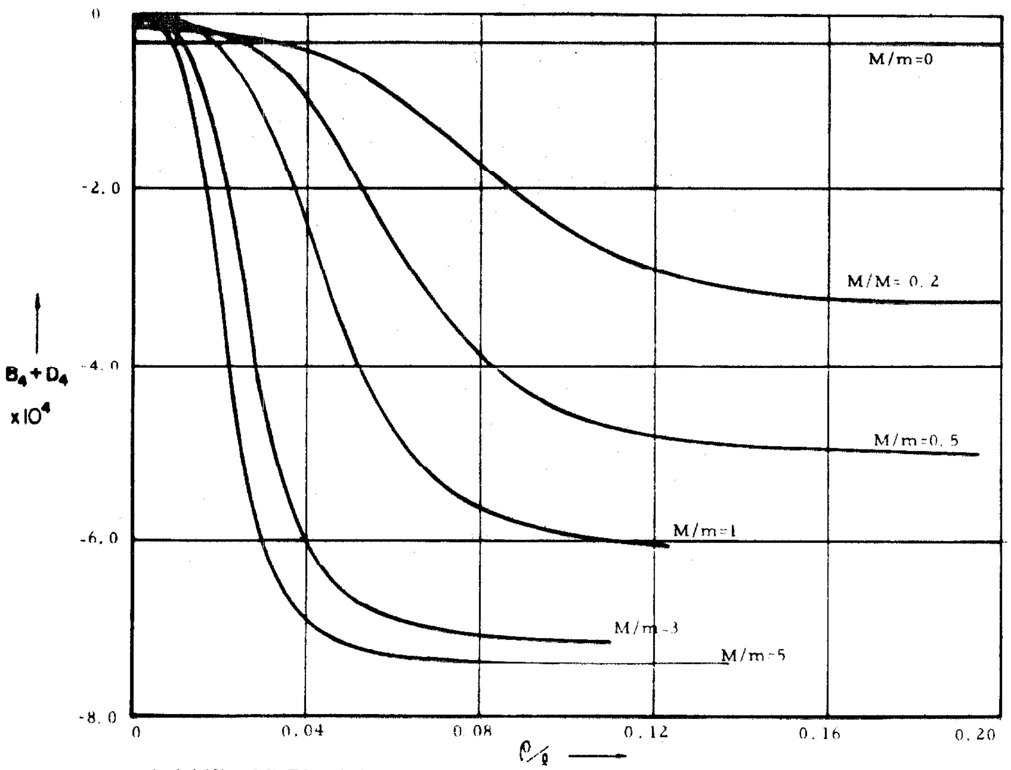
A-3-c. FIXED-FREE - THIRD MODE - MODE PARTICIPATION FACTOR



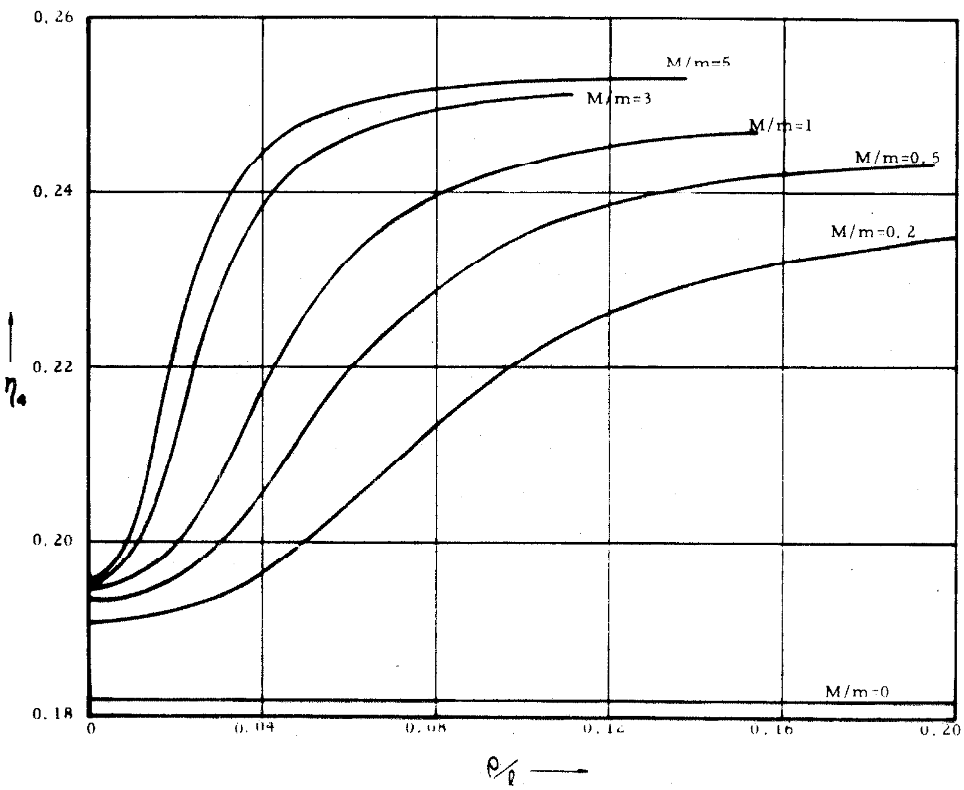
A-4-a. FIXED-FREE - FOURTH MODE - FREQUENCY EQUATION



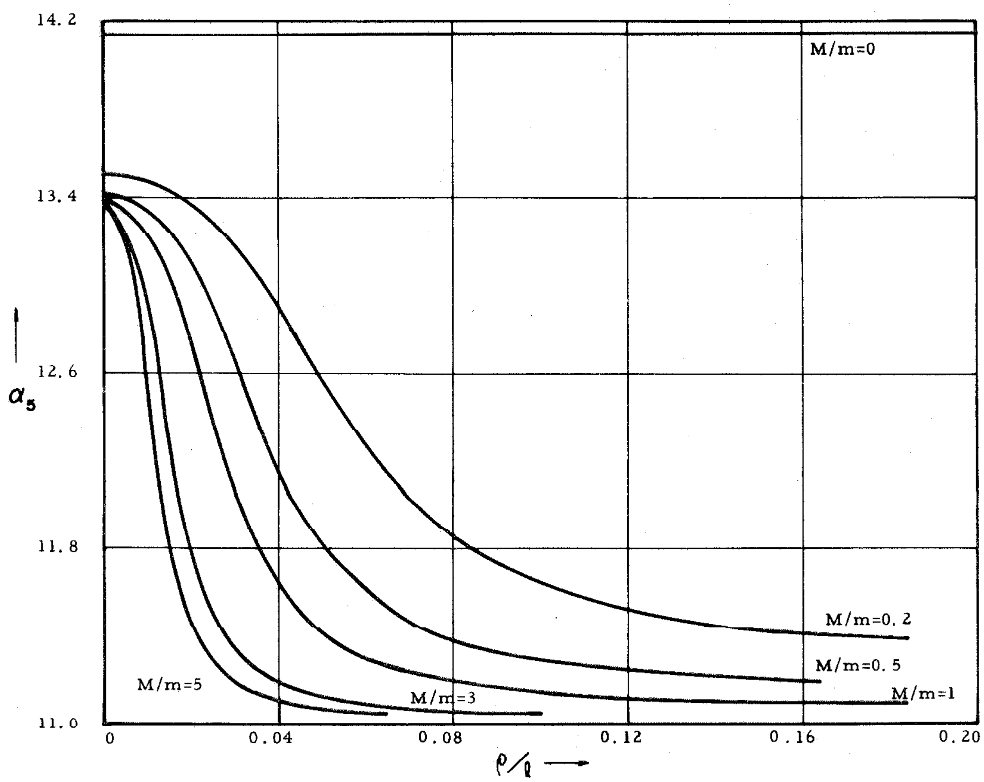
A-4-b(1). FIXED-FREE - FOURTH MODE - MODE SHAPE FACTOR



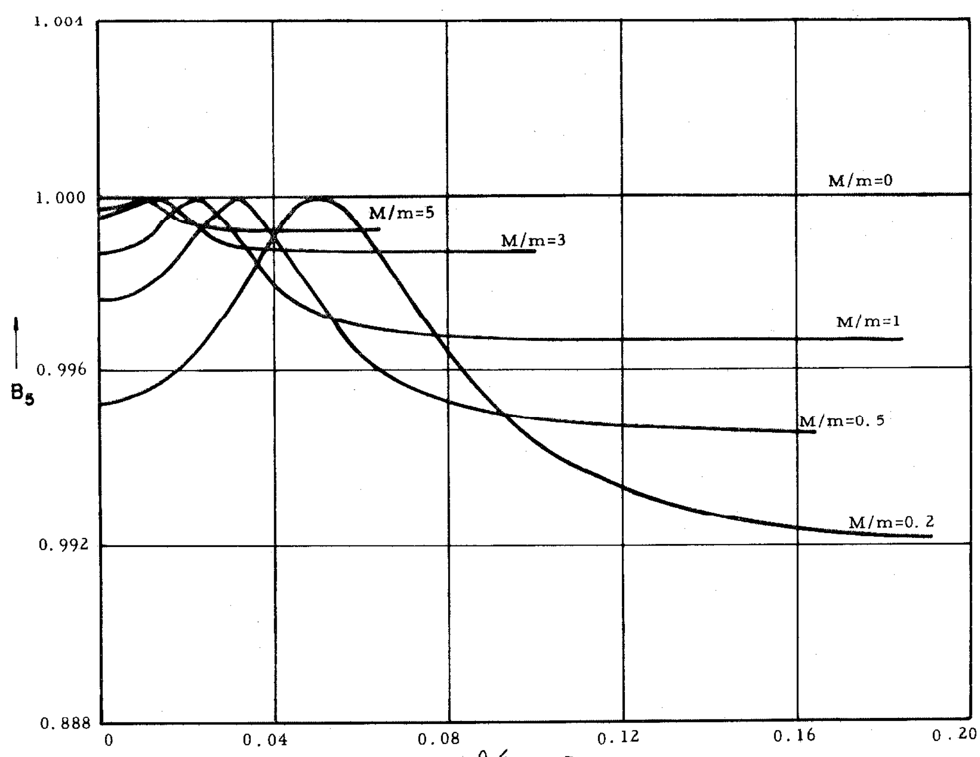
A-4-b(2). FIXED-FREE - FOURTH MODE - MODE SHAPE FACTOR



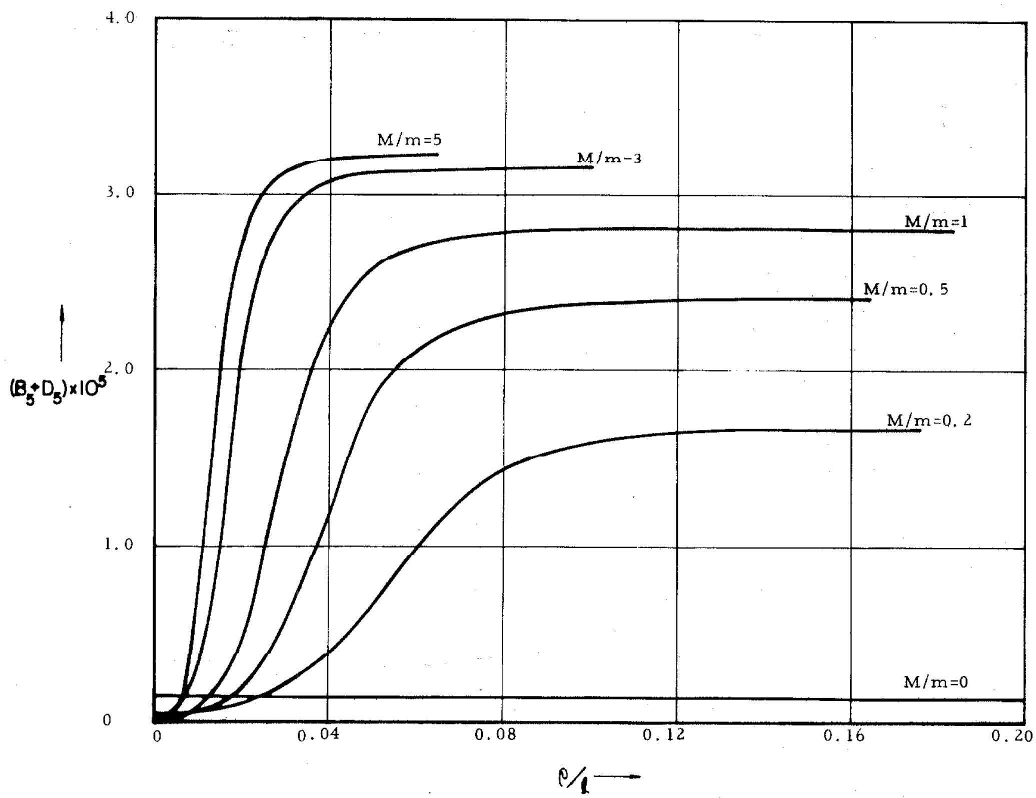
A-4-c. FIXED - FREE - FOURTH MODE - MODE PARTICIPATION FACTOR



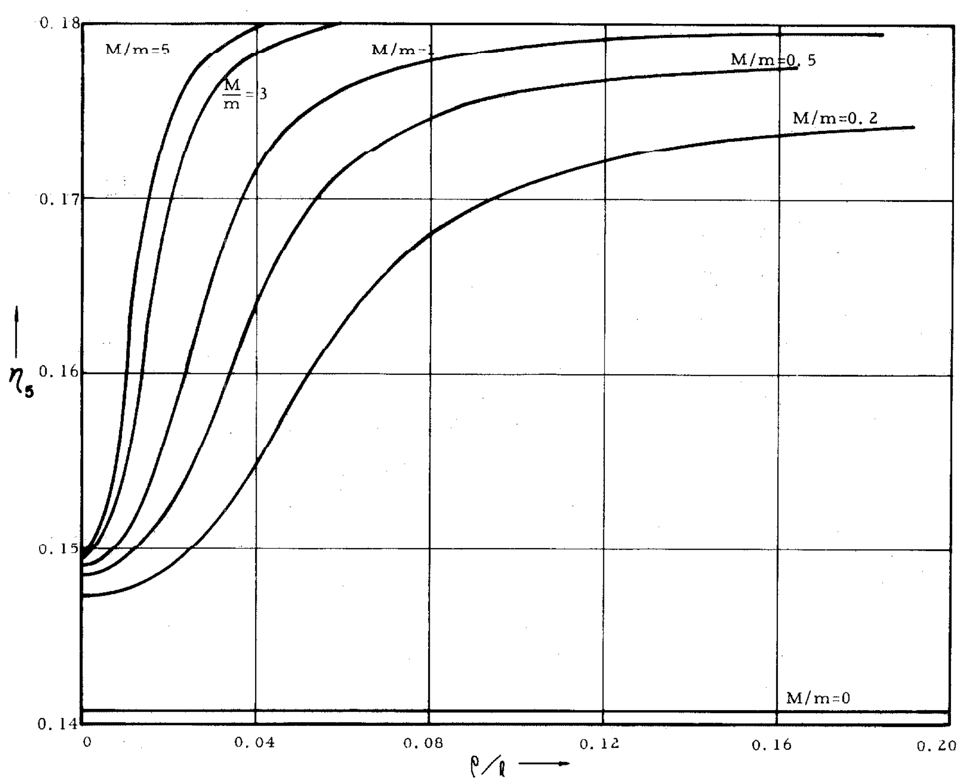
A-5-a. FIXED-FREE - FIFTH MODE - FREQUENCY ROOT



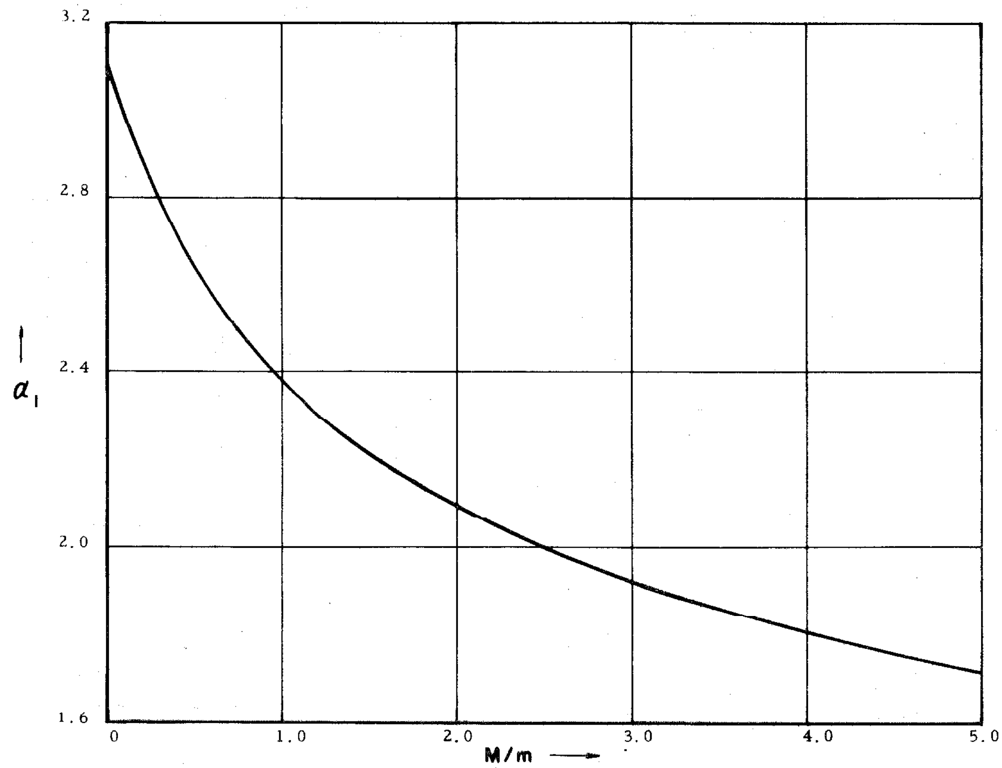
A-5-b(1). FIXED-FREE- FIFTH MODE - MODE SHAPE FACTOR



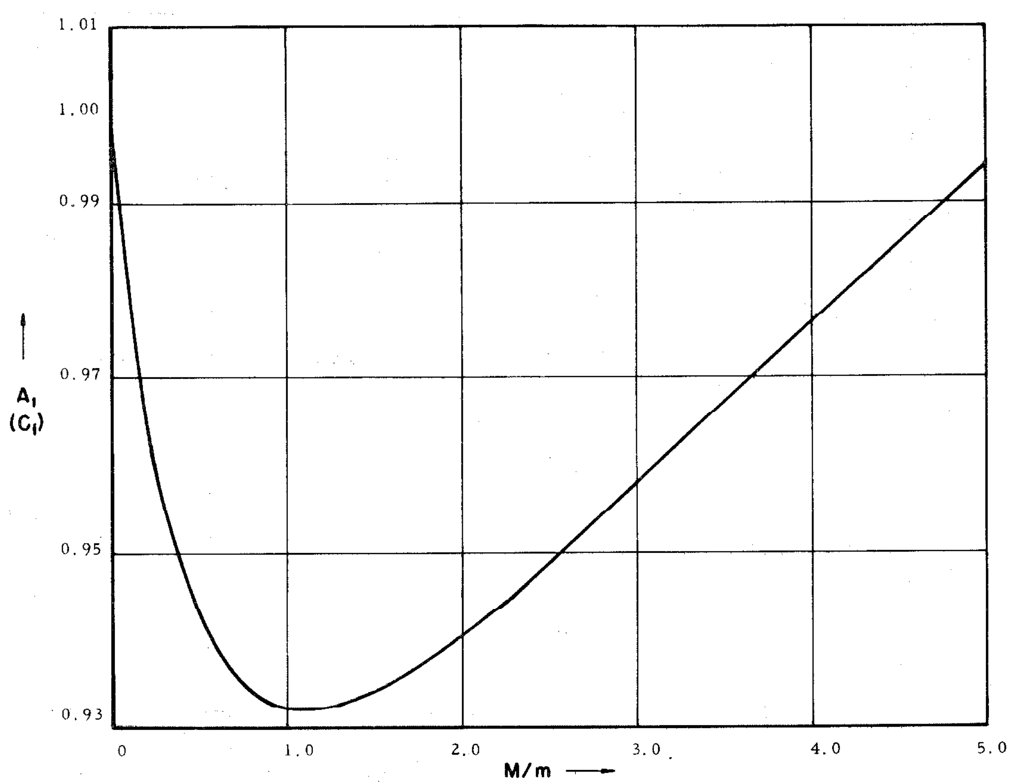
A-5-b(2). FIXED-FREE - FIFTH MODE - MODE SHAPE FACTOR



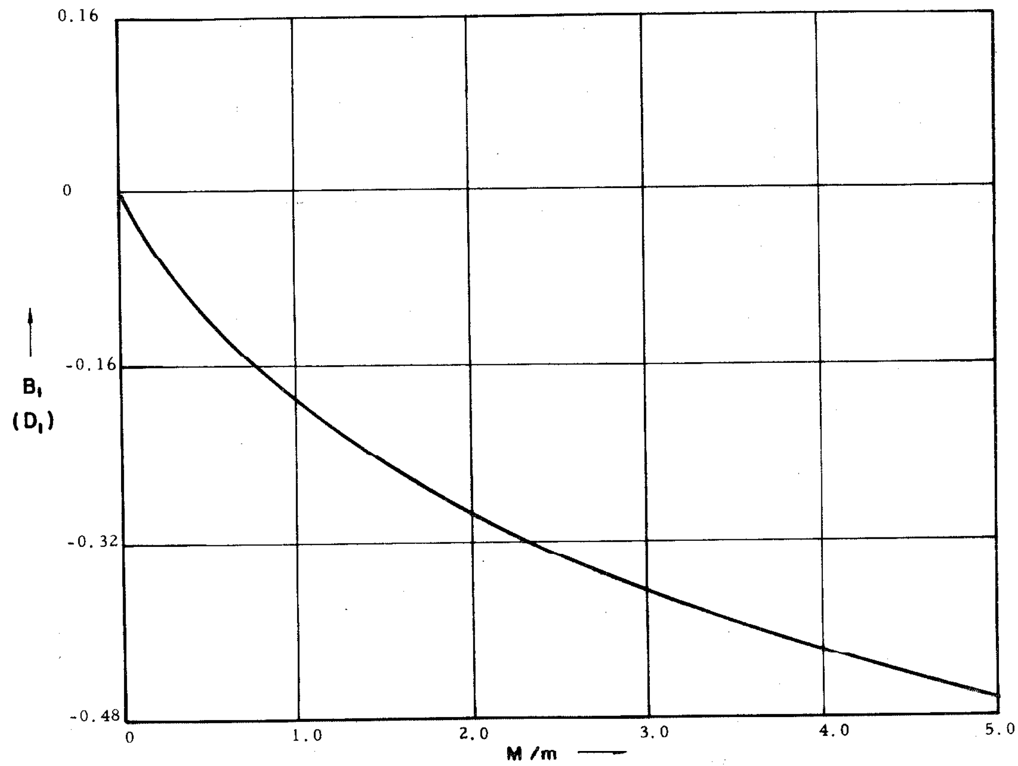
A-5-c. FIXED-FREE - FIFTH MODE - MODE PARTICIPATION FACTOR



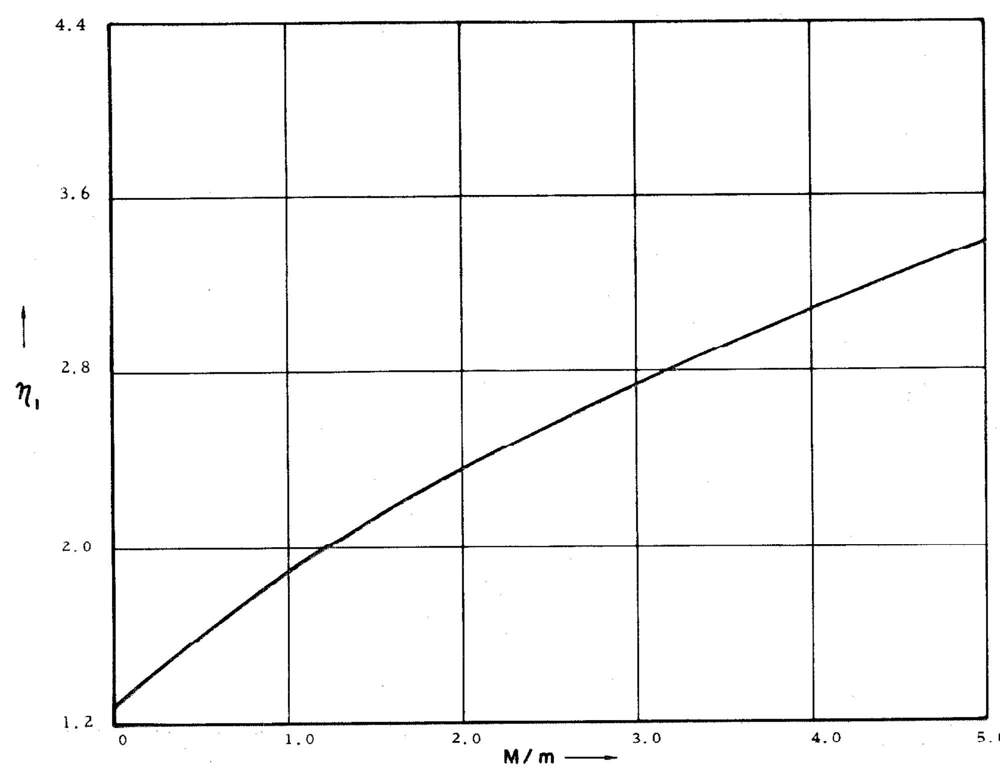
B-1-a. HINGED-HINGED, $\ell_1 = \ell/2$ - FIRST MODE - FREQUENCY ROOT



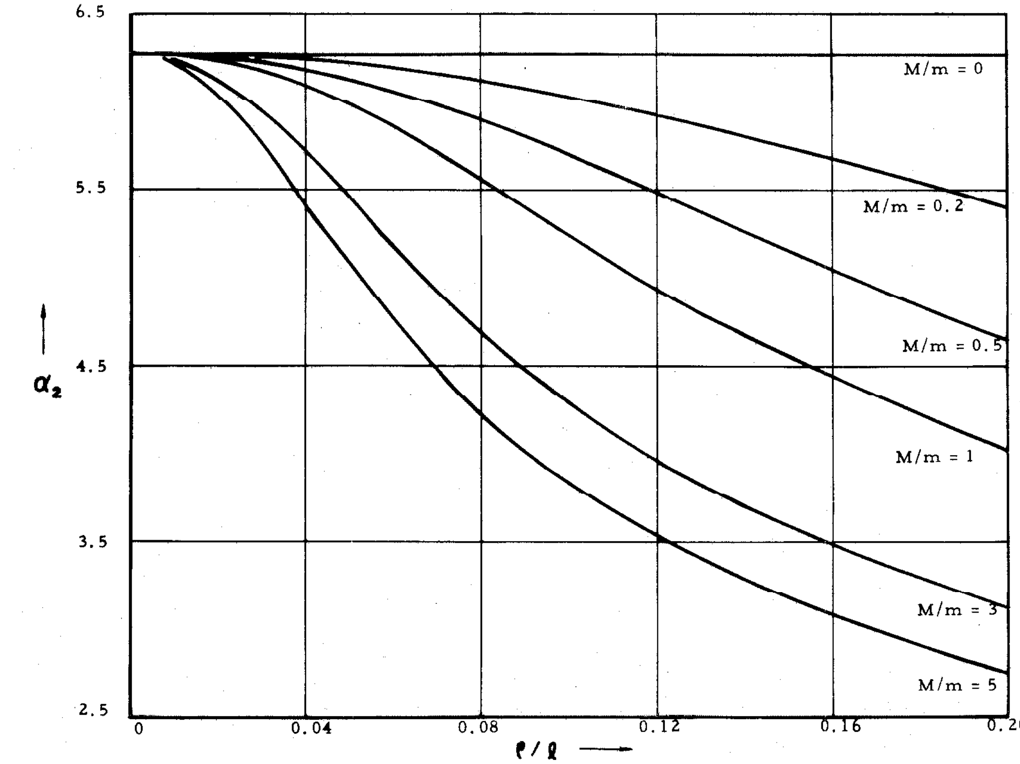
B-1-b(1). HINGED-HINGED, $\ell_1 = \ell/2$- FIRST MODE - MODE SHAPE FACTOR



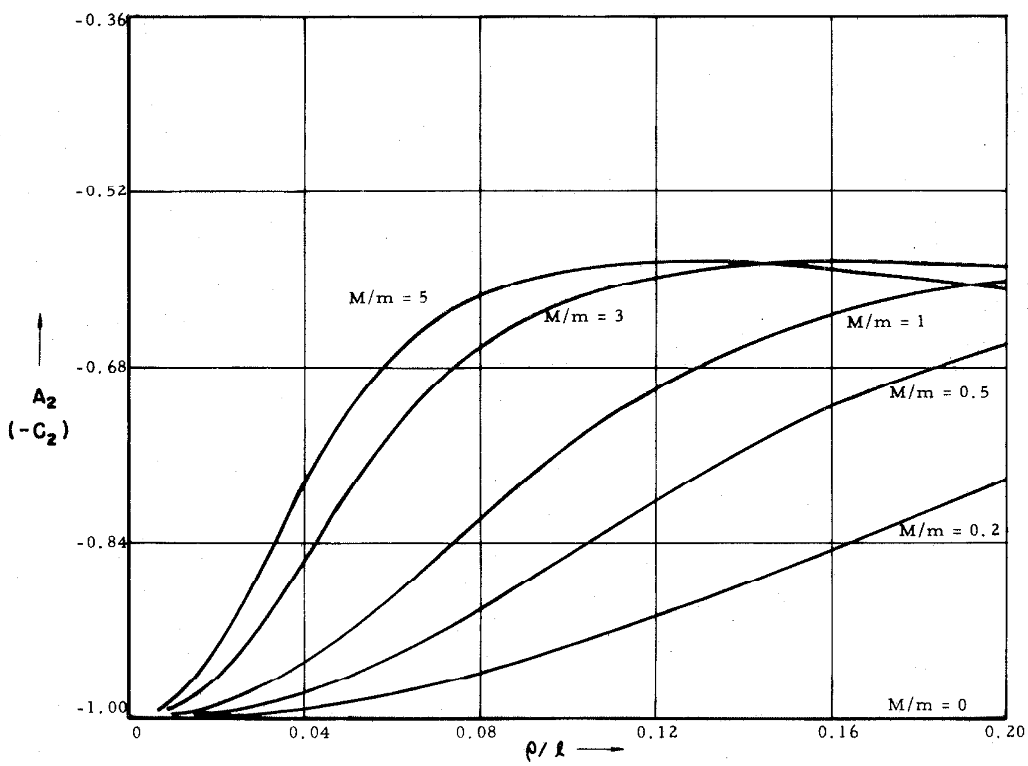
B-1-b(2). HINGED-HINGED, $\ell_1 = \ell/2$- FIRST MODE - MODE SHAPE FACTOR



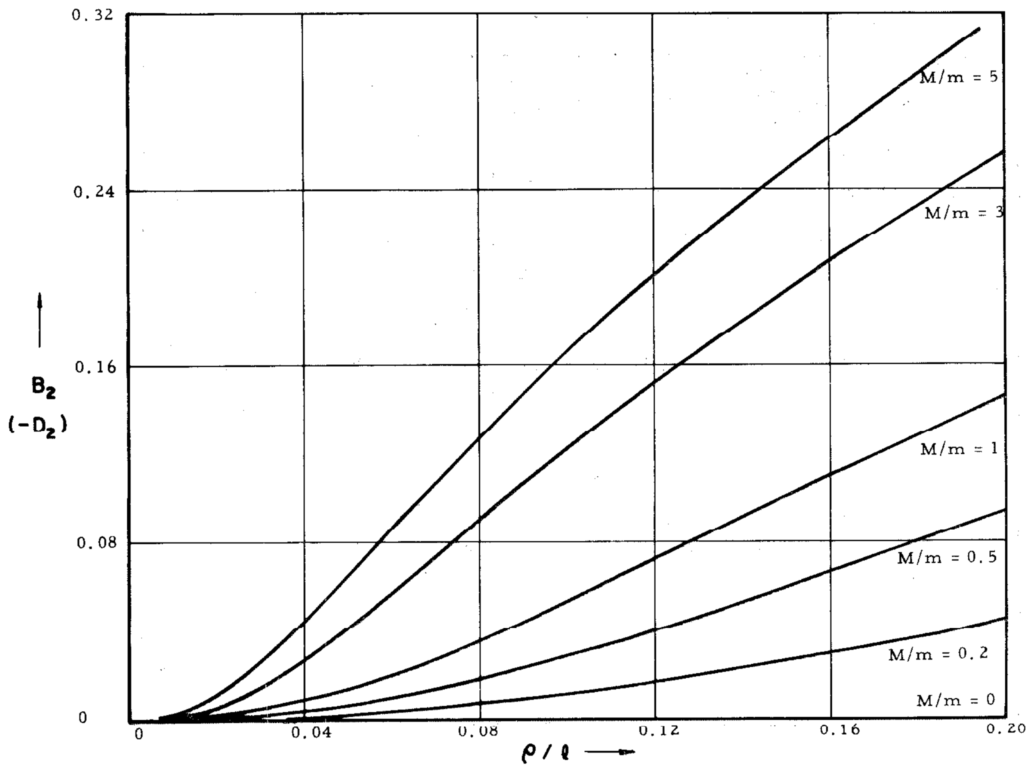
B-1-c. HINGED-HINGED, $\ell_1 = \ell/2$- FIRST MODE - MODE PARTICIPATION FACTOR



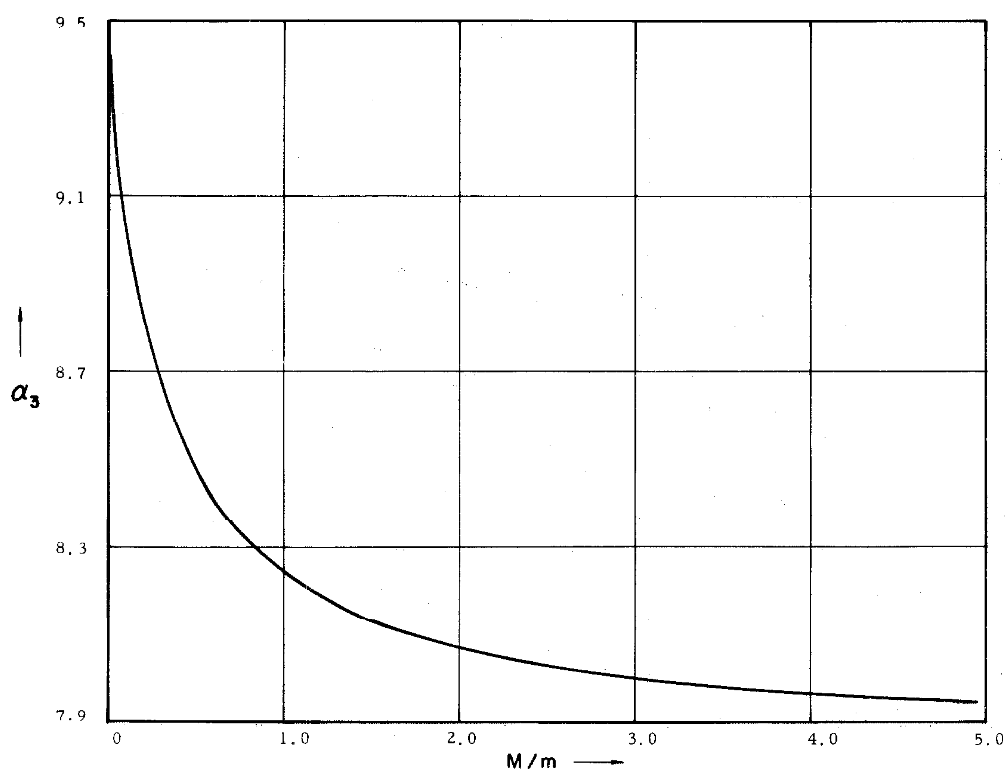
B-2-a. HINGED-HINGED, $\ell_1 = \ell/2$ - SECOND MODE - FREQUENCY ROOT



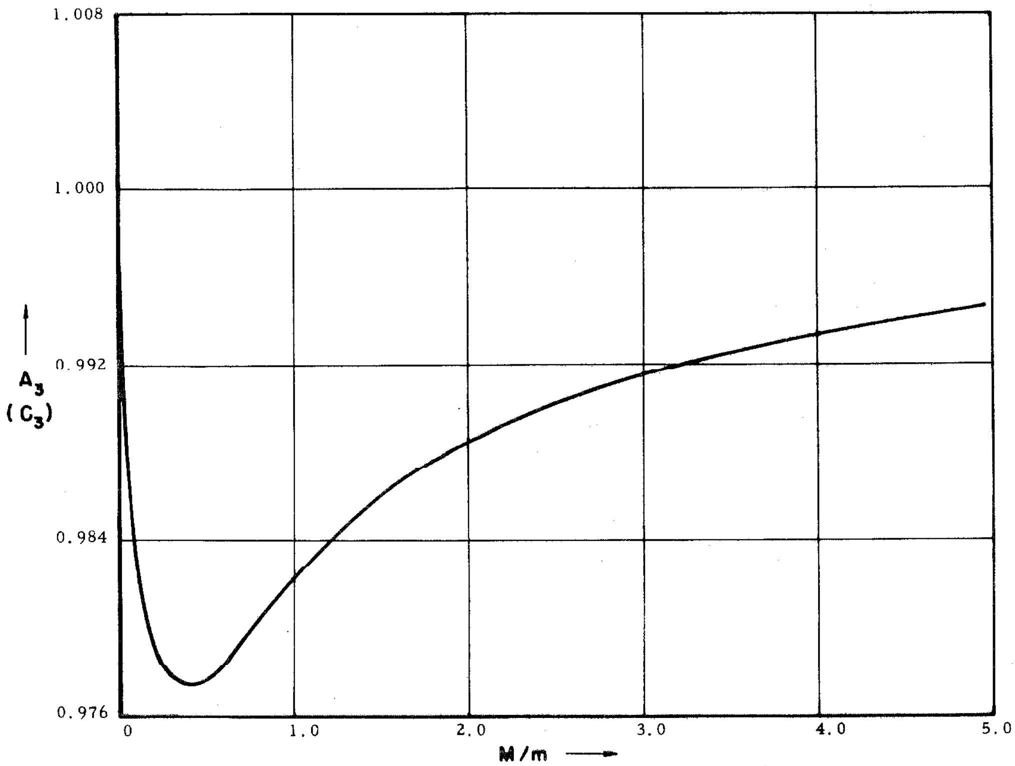
B-2-b(1). HINGED-HINGED, $\ell_1 = \ell/2$ - SECOND MODE - MODE SHAPE FACTOR



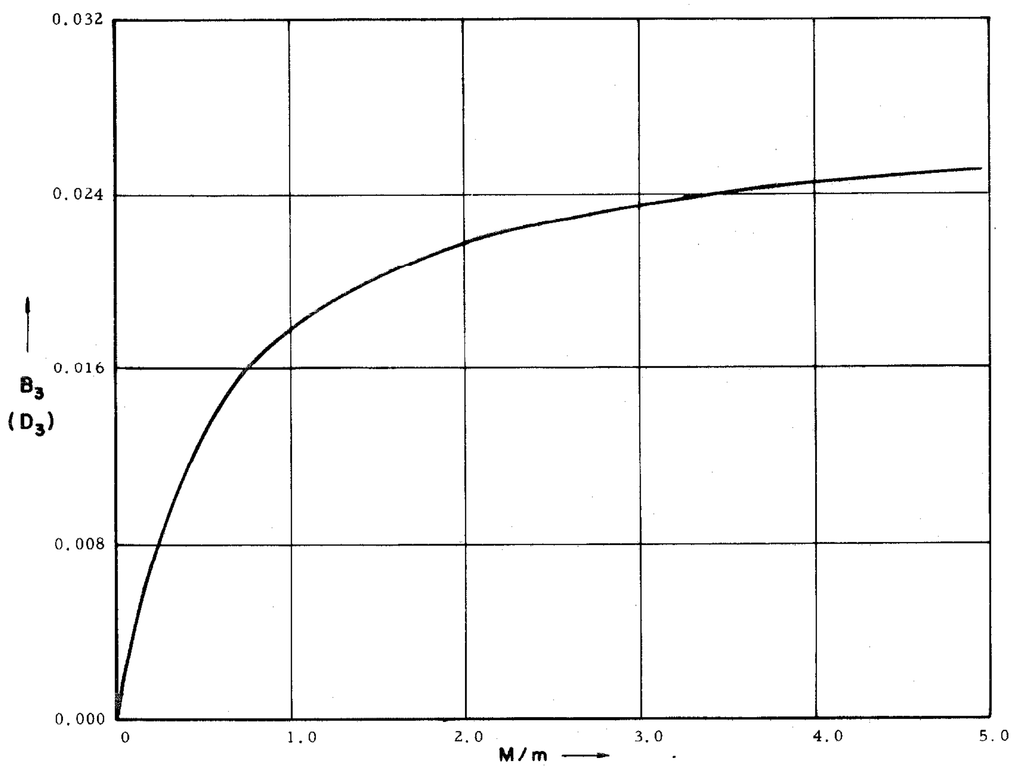
B-2-b(2). HINGED-HINGED, $\ell_1 = \ell/2$ - SECOND MODE - MODE SHAPE FACTOR



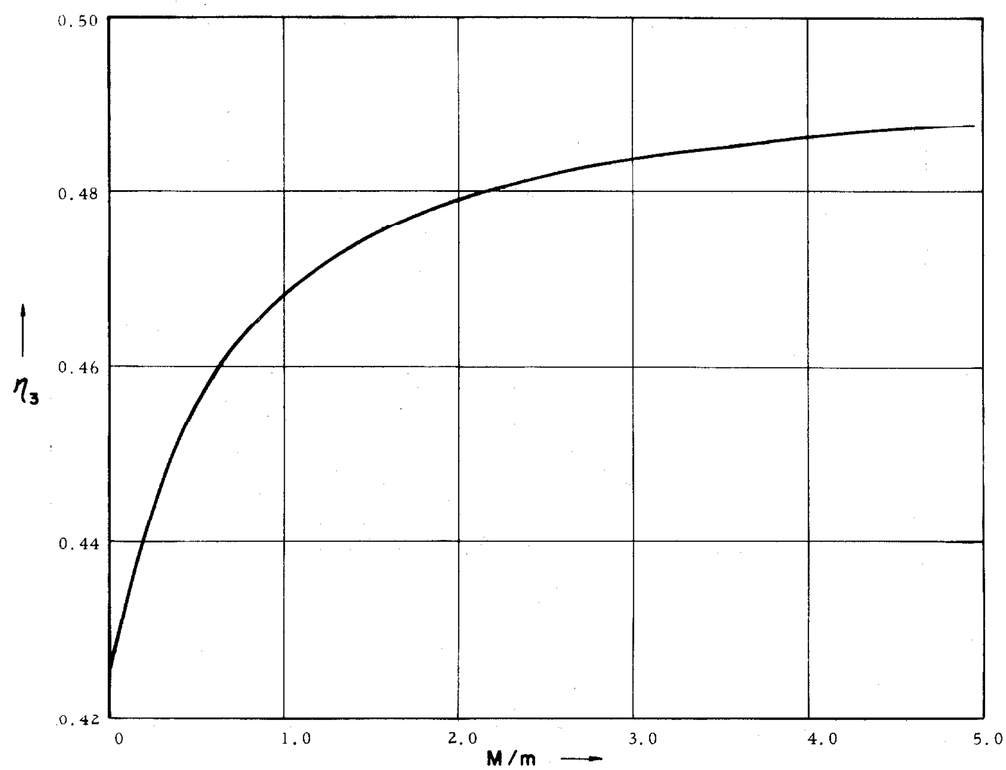
B-3-a. HINGED-HINGED, $\ell_1 = \ell/2$ - THIRD MODE - FREQUENCY ROOT



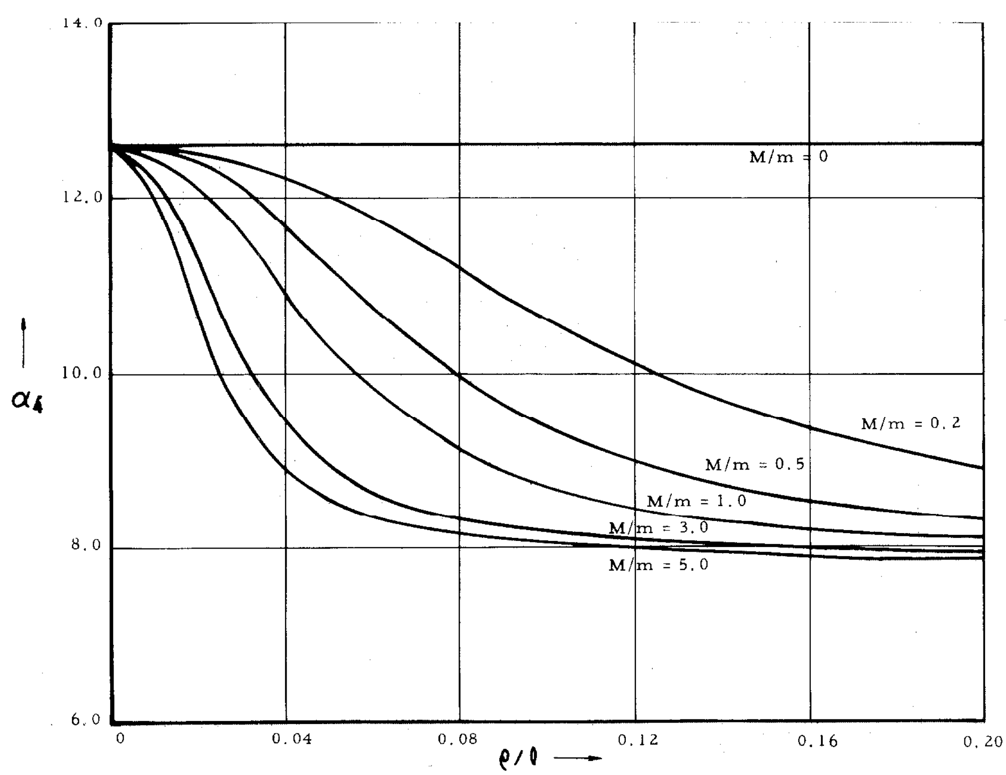
B-3-b(1). HINGED-HINGED, $\ell_1 = \ell/2$ - THIRD MODE - MODE SHAPE FACTOR



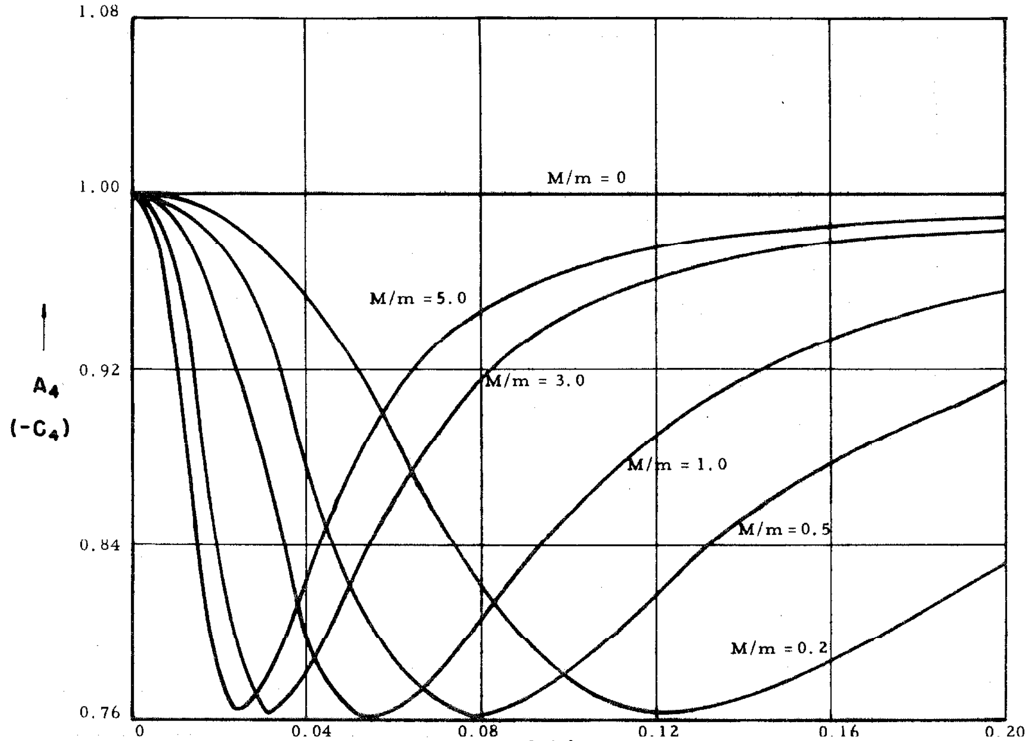
B-3-b(2). HINGED-HINGED, $\ell_1 = \ell/2$ - THIRD MODE - MODE SHAPE FACTOR



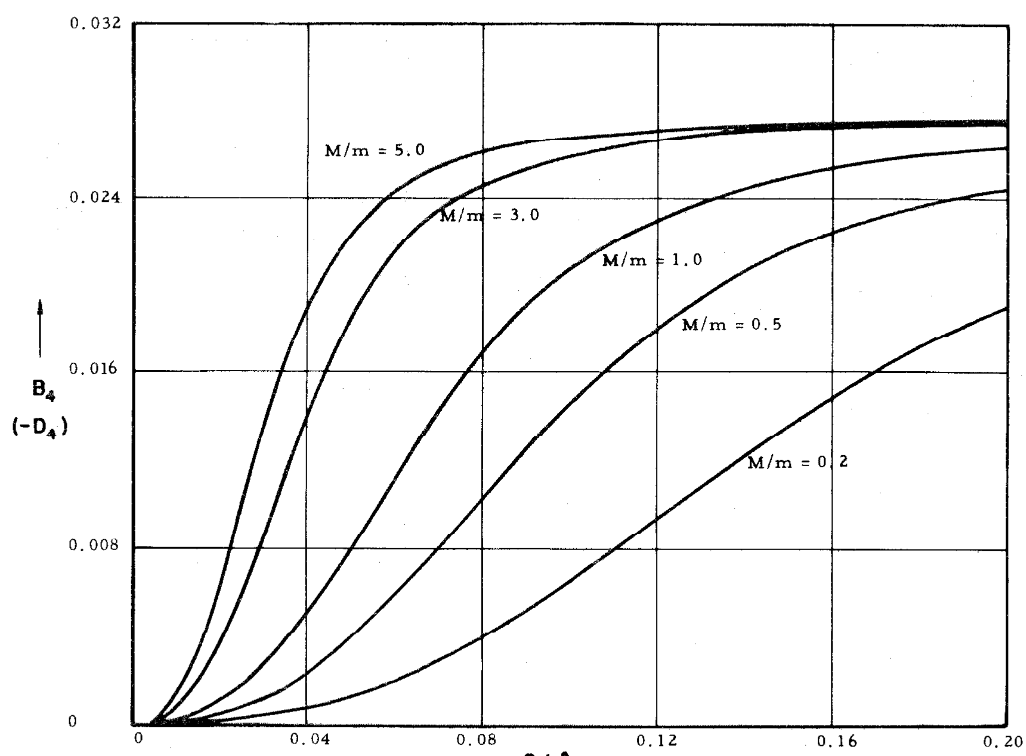
B-3-c. HINGED-HINGED, $\ell_1 = \ell/2$ - THIRD MODE - MODE PARTICIPATION FACTOR



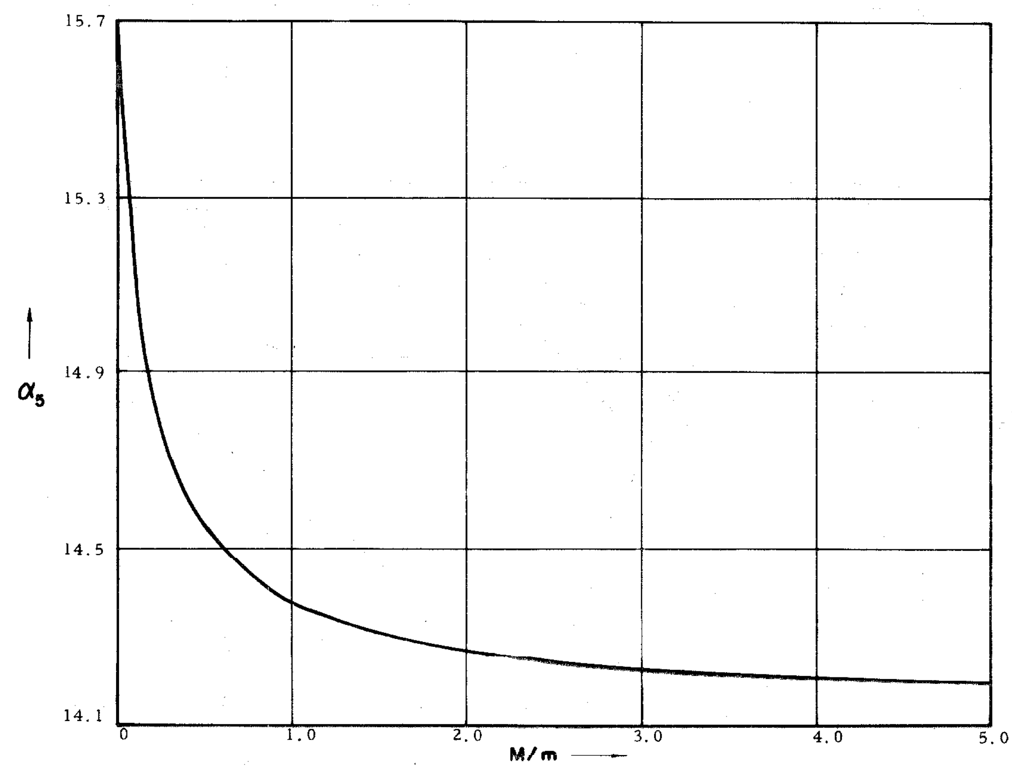
B-4-a. HINGED-HINGED, $\ell_1 = \ell/2$ - FOURTH MODE - FREQUENCY ROOT



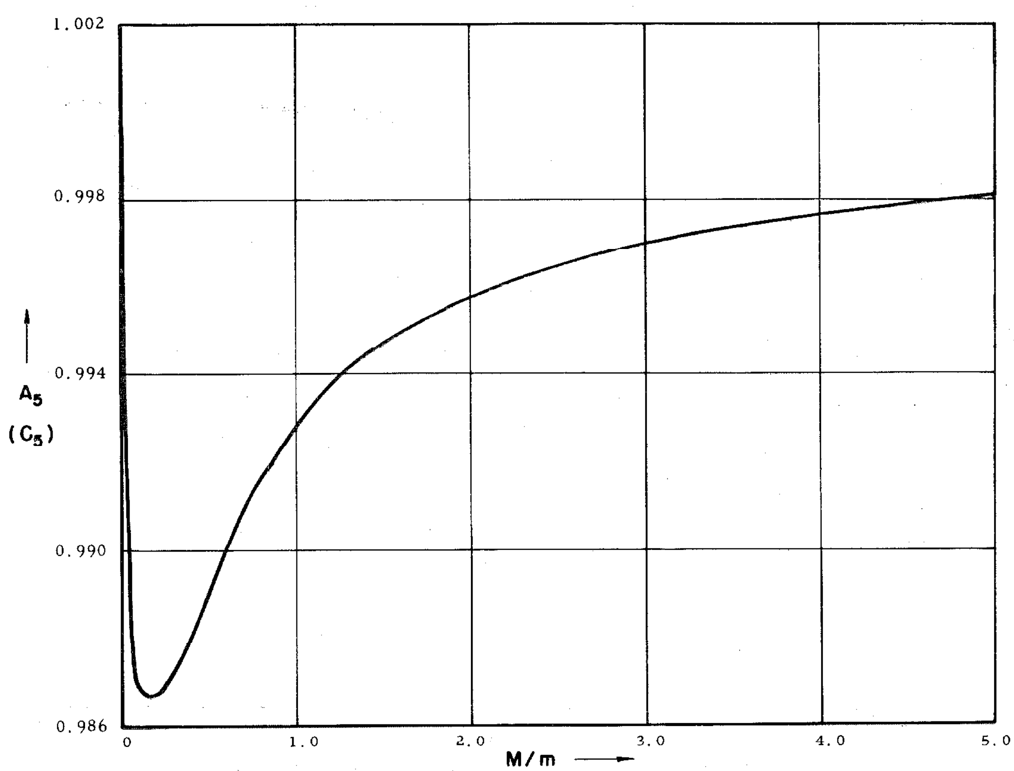
B-4-b(1). HINGED-HINGED, $\ell_1 = \ell/2$ - FOURTH MODE -MODE SHAPE FACTOR



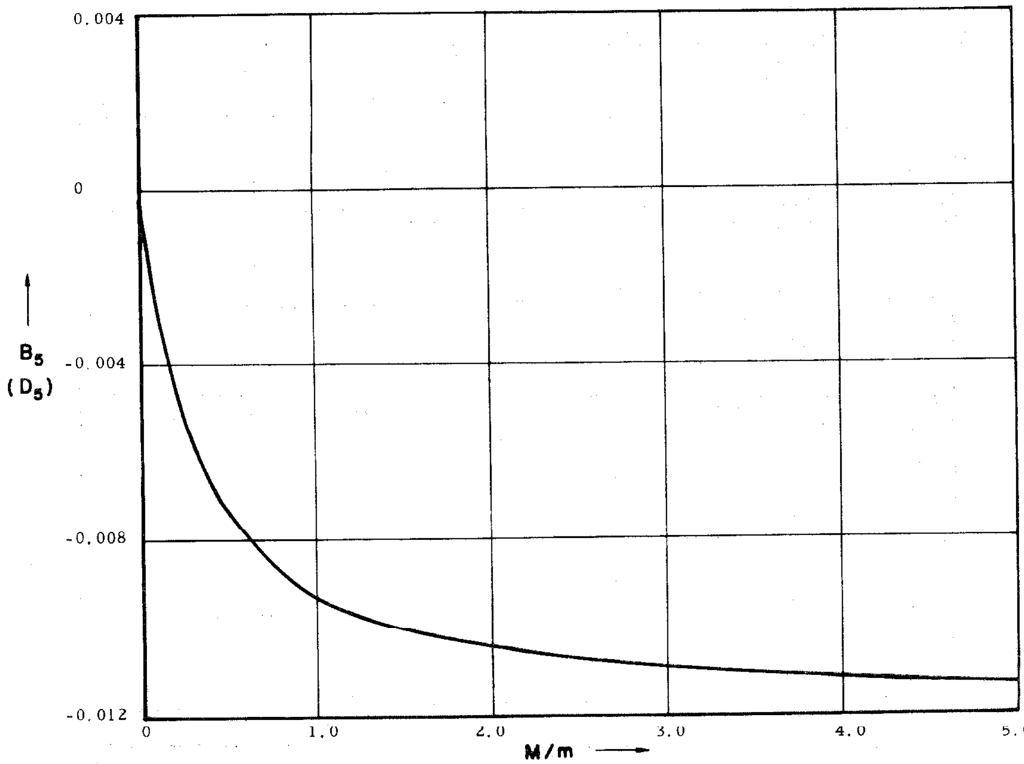
B-4-b(2). HINGED-HINGED, $\ell_1 = \ell/2$ - FOURTH MODE - MODE SHAPE FACTOR



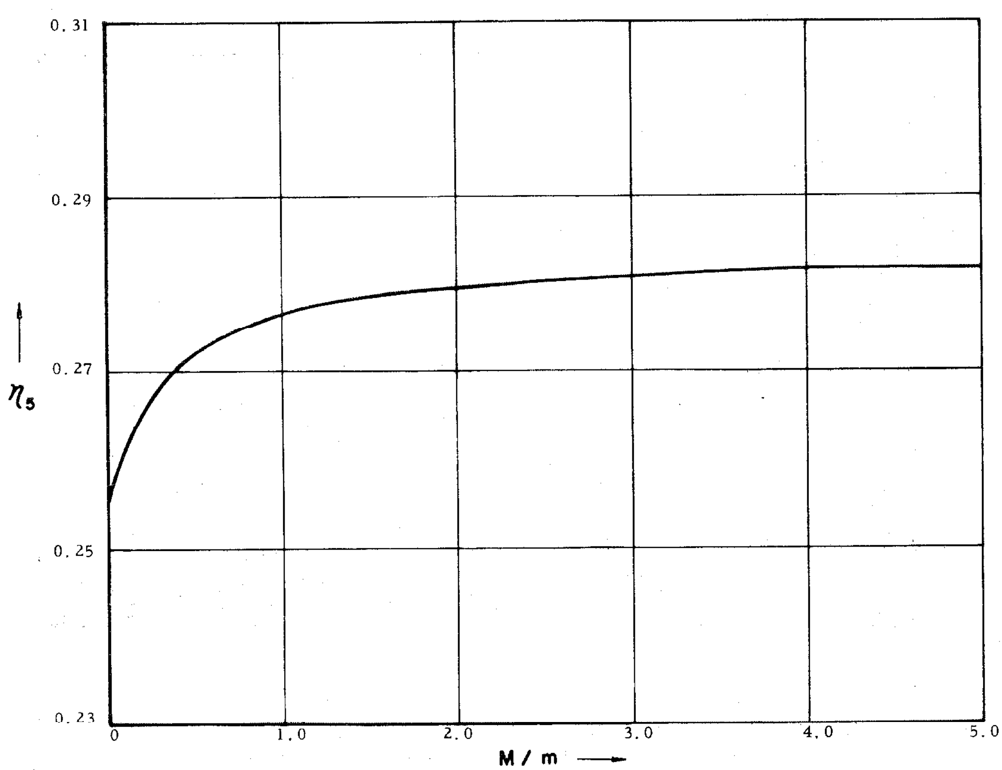
B-5-a. HINGED-HINGED, $\ell_1 = \ell/2$ - FIFTH MODE - FREQUENCY ROOT



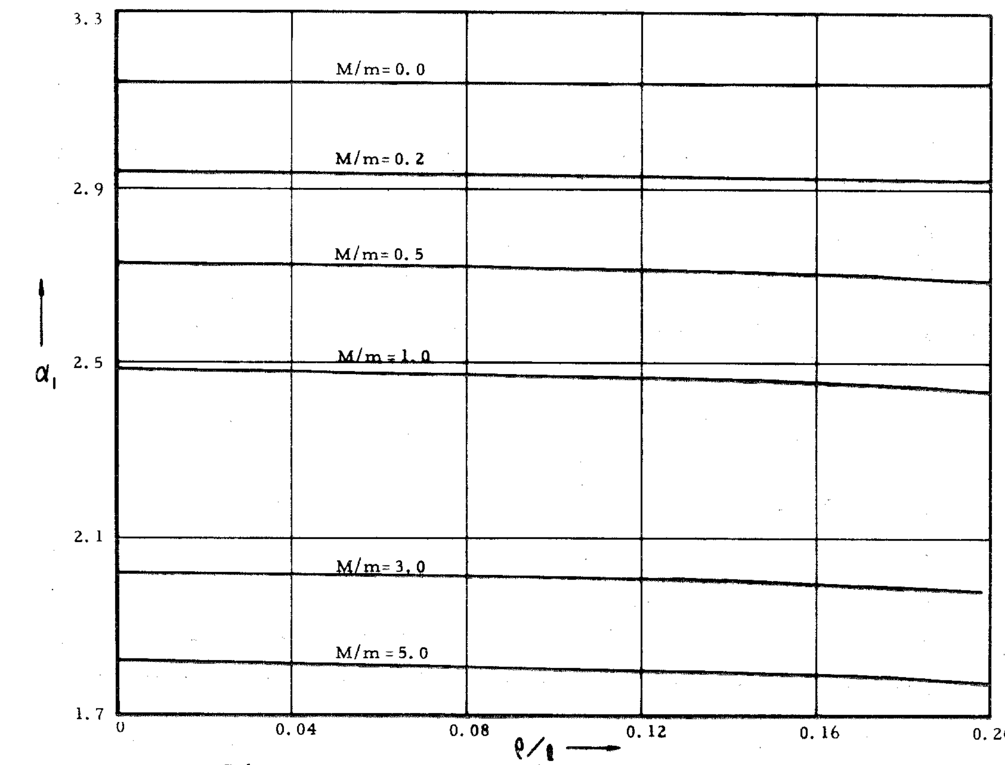
B-5-b(1). HINGED-HINGED, $\ell_1 = \ell/2$ - FIFTH MODE - MODE SHAPE FACTOR



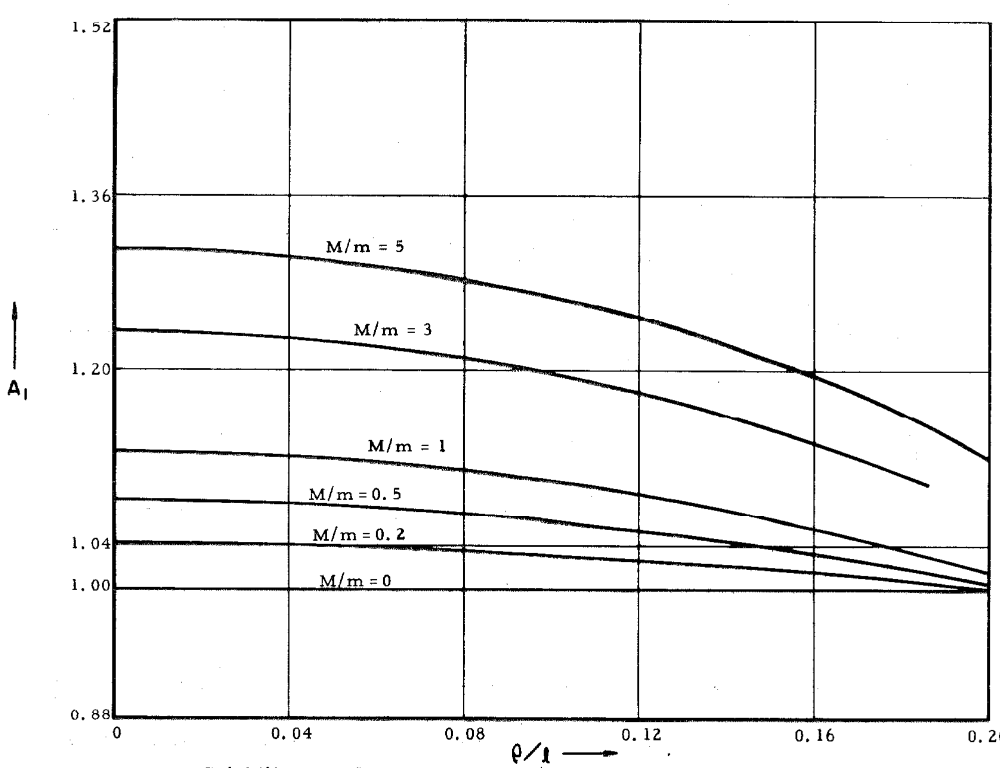
B-5-b(2). HINGED-HINGED, $\ell_1 = \ell/2$ - FIFTH MODE - MODE SHAPE FACTOR



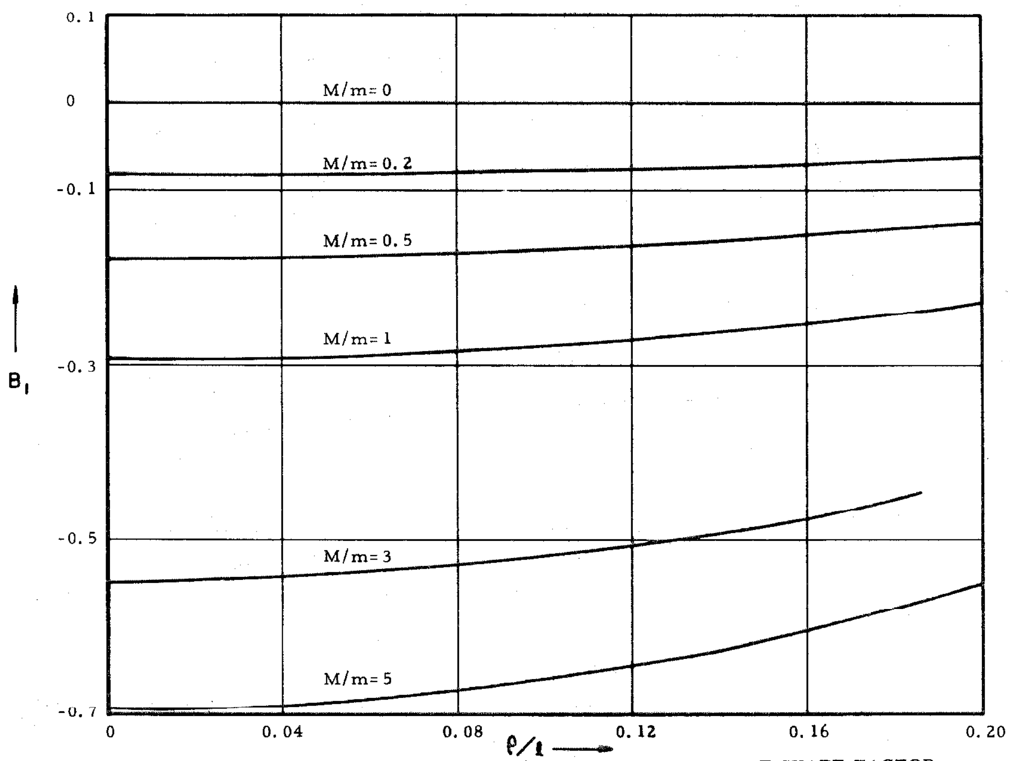
B-5-c. HINGED-HINGED, $\ell_1 = \ell/2$ - FIFTH MODE - MODE PARTICIPATION FACTOR



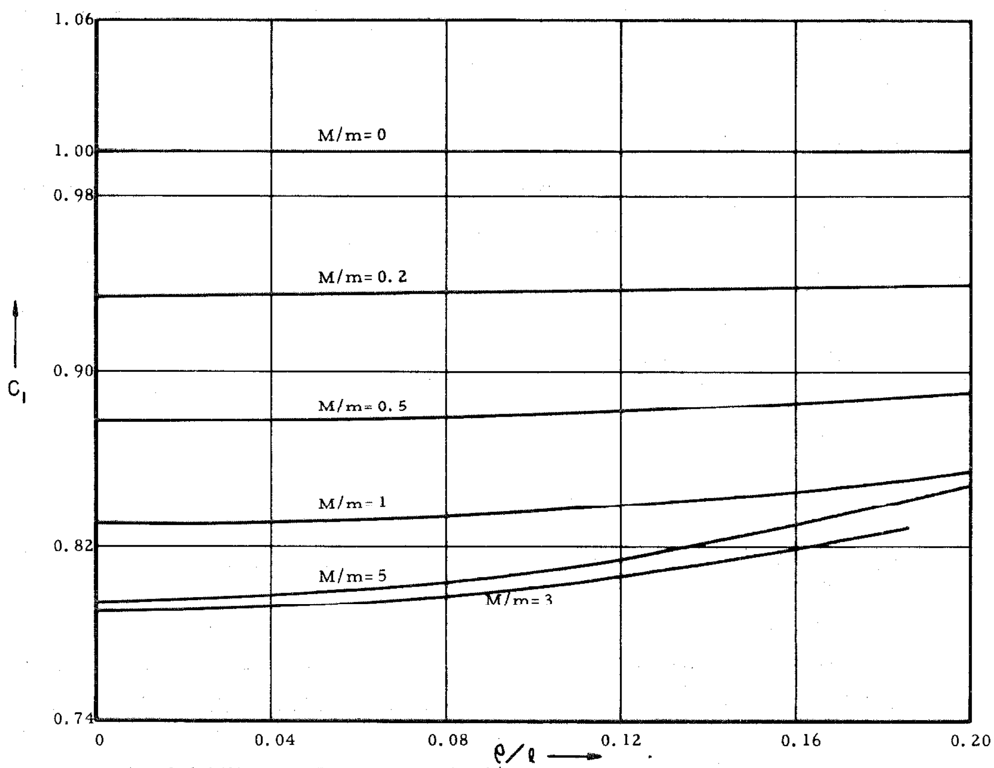
C-1-a. HINGED-HINGED, $\ell_1 = \ell/3$ - FIRST MODE - FREQUENCY ROOT



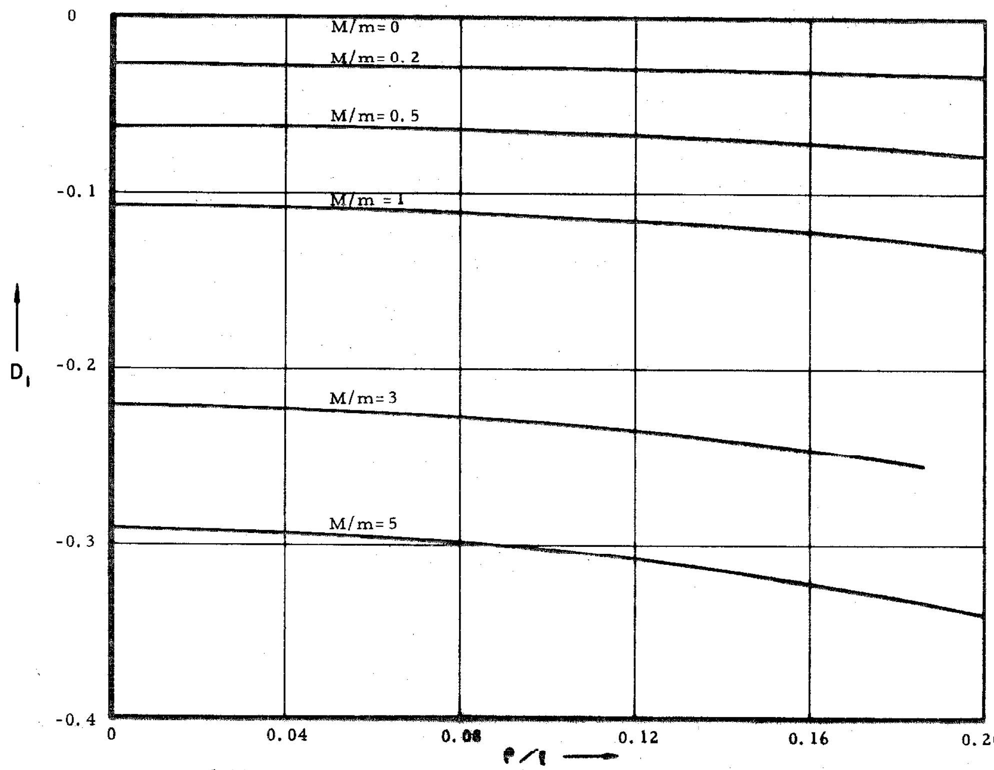
C-1-b(1). HINGED-HINGED, $\ell_1 = \ell/3$ - FIRST MODE - MODE SHAPE FACTOR



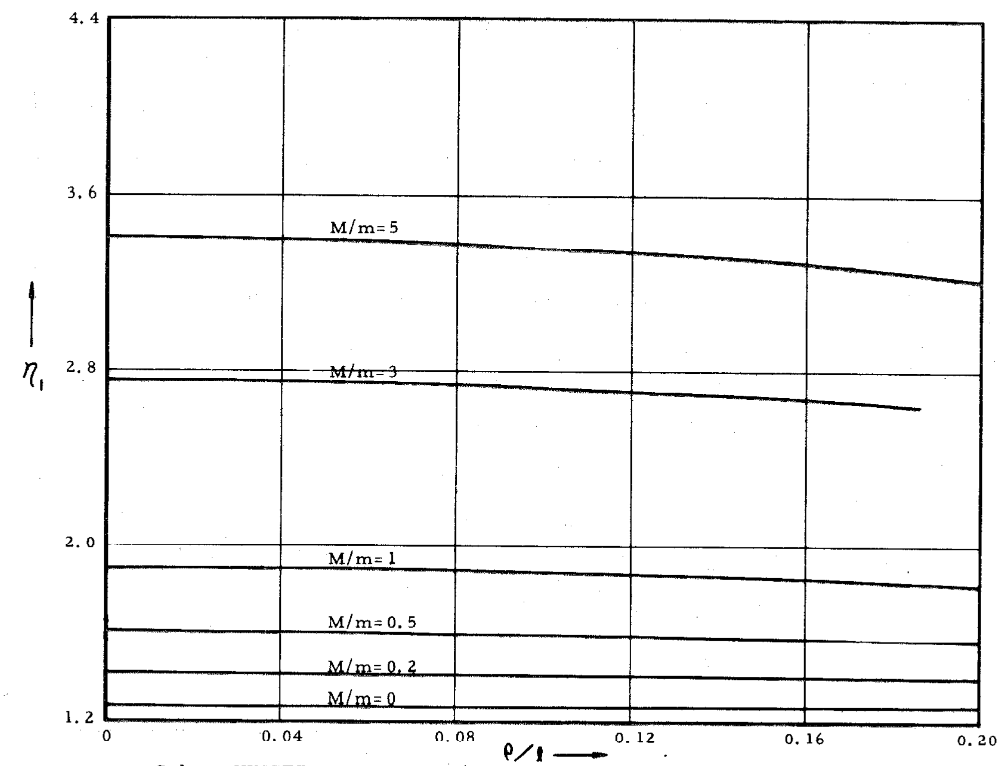
C-1-b(2). HINGED-HINGED, $\ell_1 = \ell/3$ - FIRST MODE - MODE SHAPE FACTOR



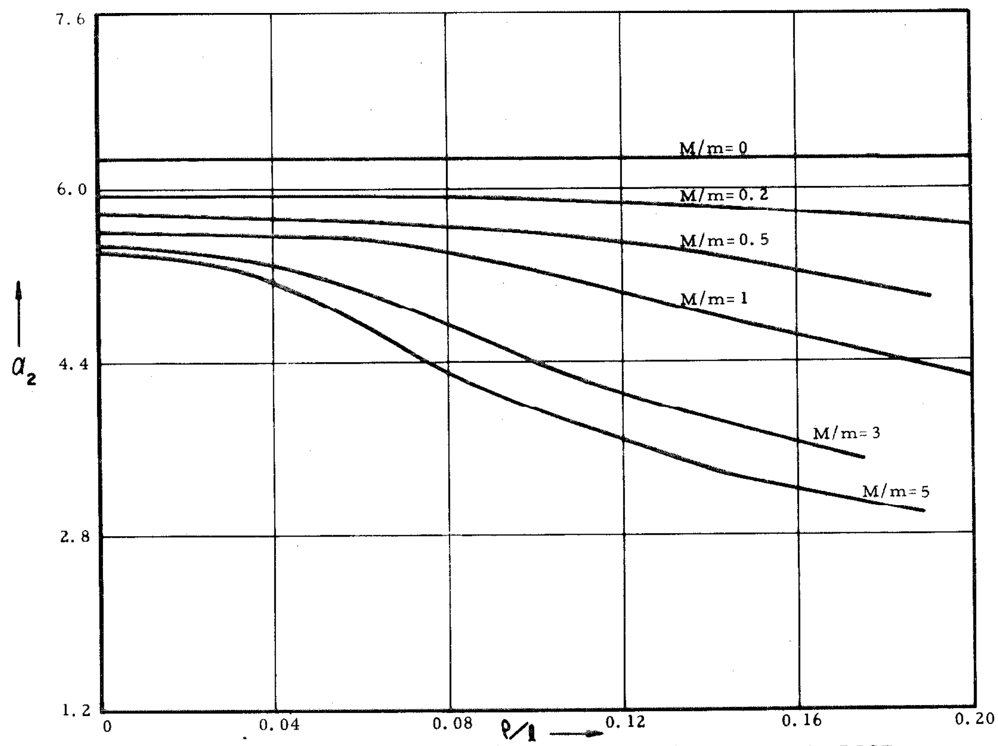
C-1-b(3). HINGED-HINGED, $\ell_1 = \ell/3$ - FIRST MODE - MODE SHAPE FACTOR



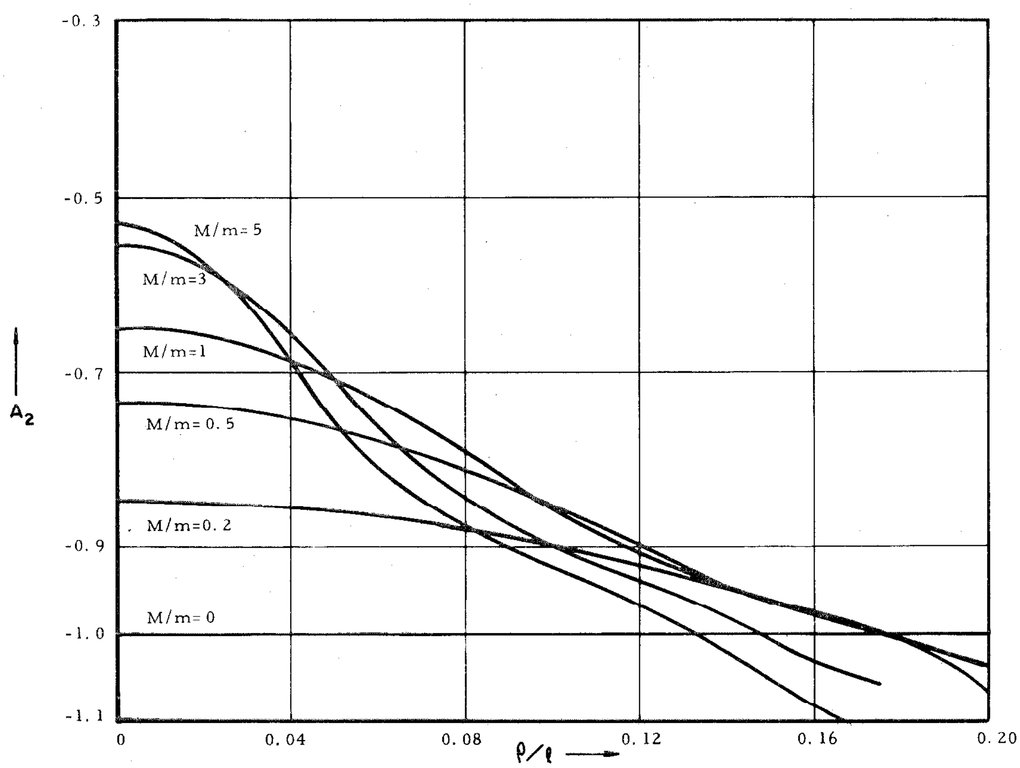
C-1-b(4). HINGED-HINGED, $l_1 = l/3$ - FIRST MODE - MODE SHAPE FACTOR



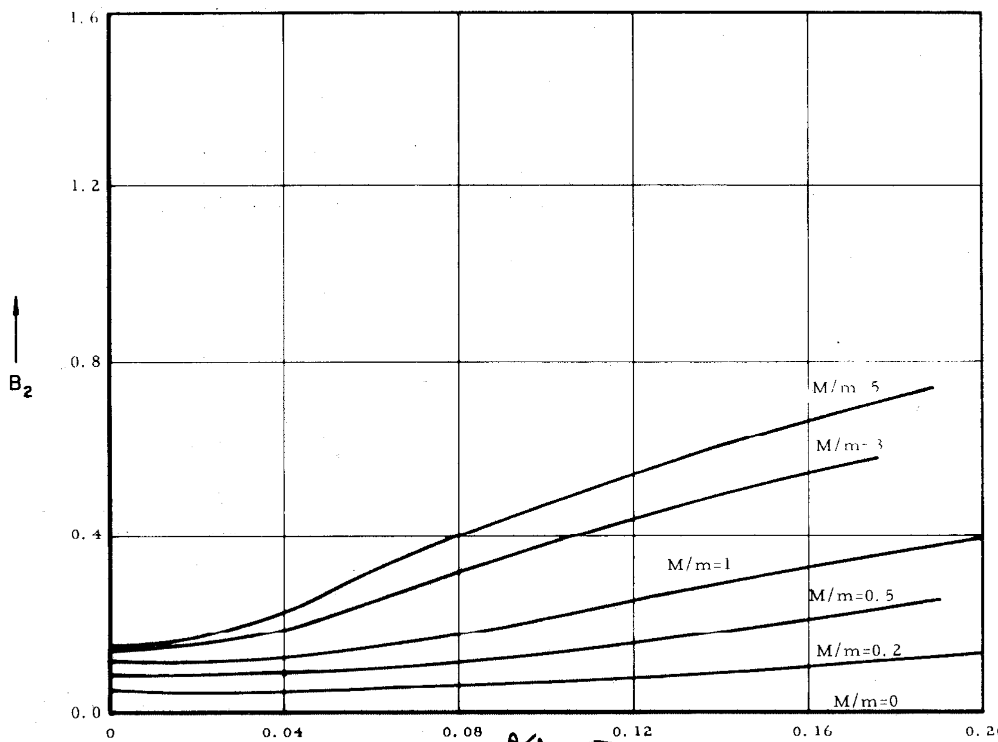
C-1-c. HINGED-HINGED, $l_1 = l/3$ - FIRST MODE - MODE PARTICIPATION FACTOR



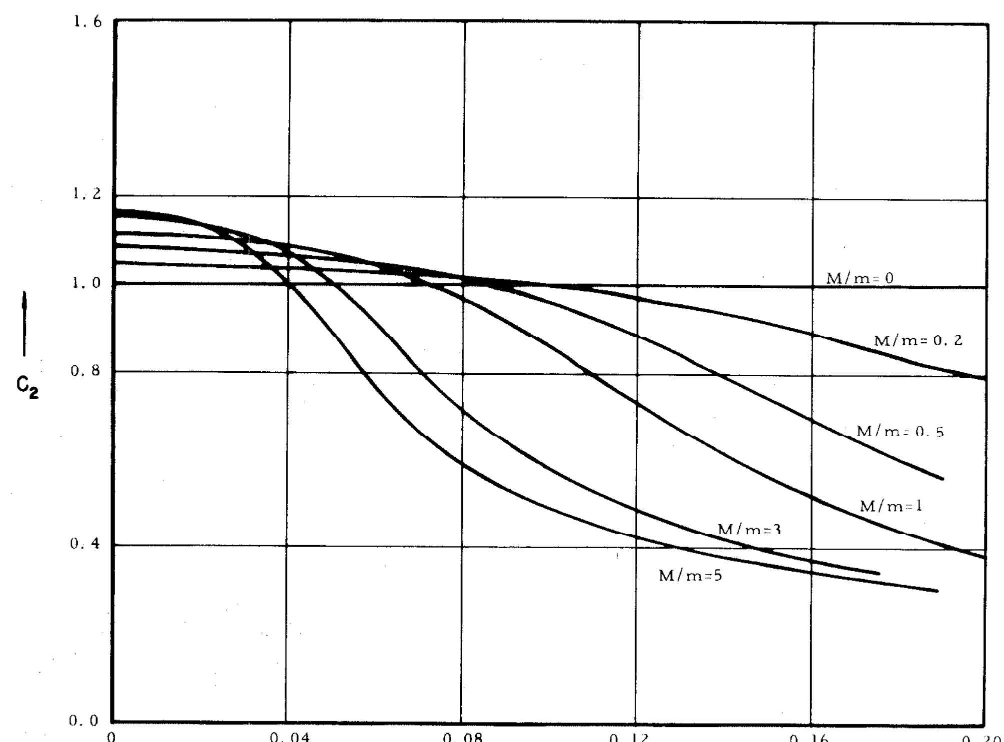
C-2-a. HINGED-HINGED, $\ell_1 = \ell/3$ - SECOND MODE - FREQUENCY ROOT



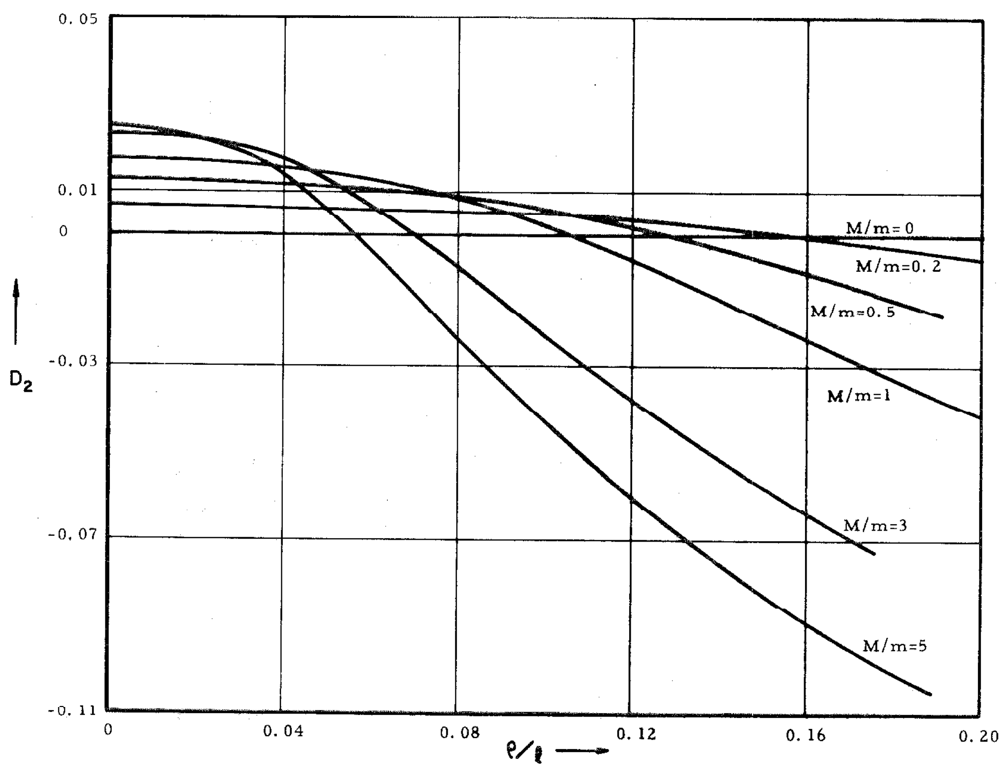
C-2-b(1). HINGED-HINGED, $\ell_1 = \ell/3$ -SECOND MODE - MODE SHAPE FACTOR



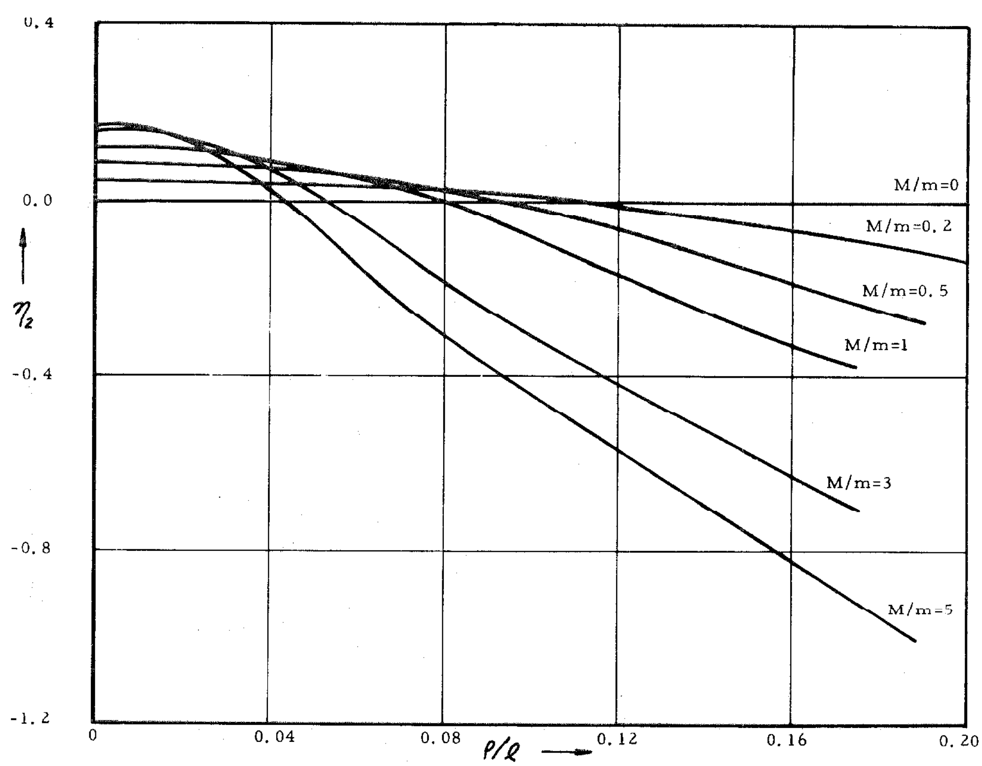
C-2-b(2). HINGED-HINGED, $\ell_1 = \ell/3$ - SECOND MODE - MODE SHAPE FACTOR



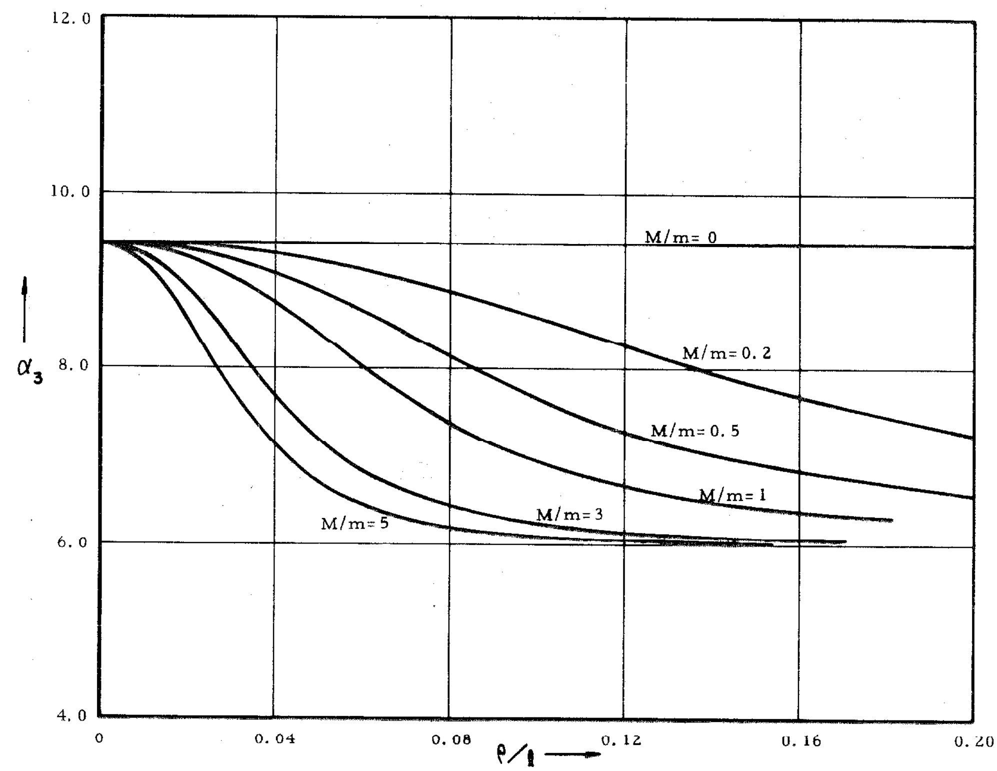
C-2-b(3). HINGED-HINGED, $\ell_1 = \ell/3$ - SECOND MODE - MODE SHAPE FACTOR



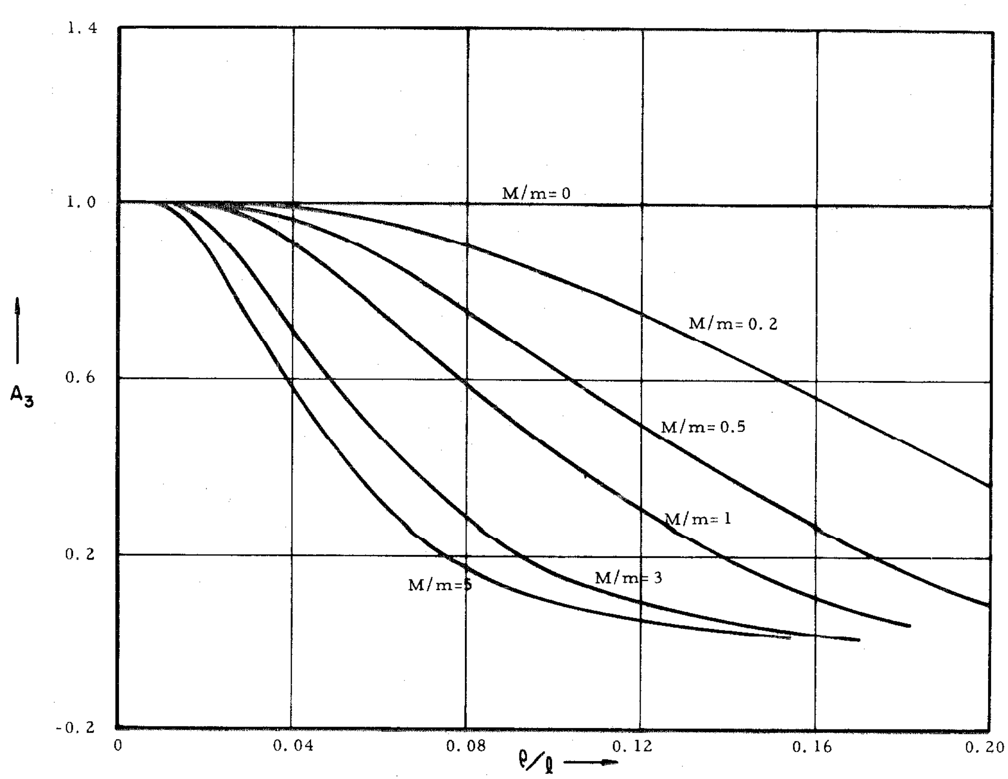
C-2-b(4). HINGED-HINGED, $\ell_1 = \ell/3$ - SECOND MODE - MODE SHAPE FACTOR



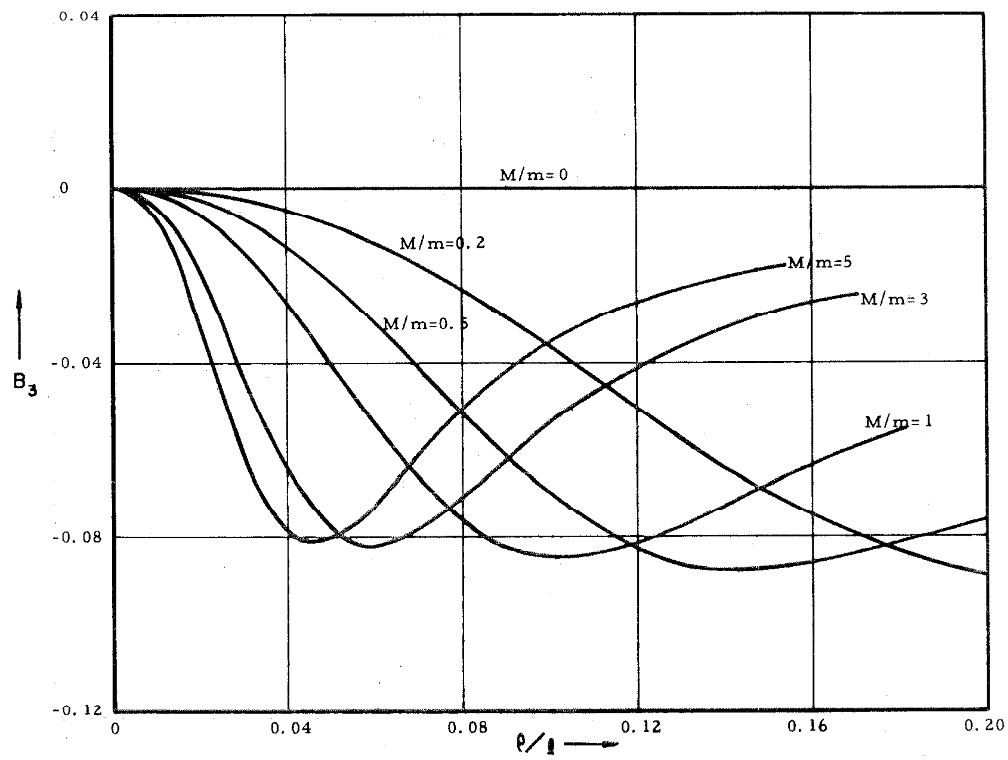
C-2-c. HINGED-HINGED, $\ell_1 = \ell/3$ - SECOND MODE - MODE PARTICIPATION FACTOR



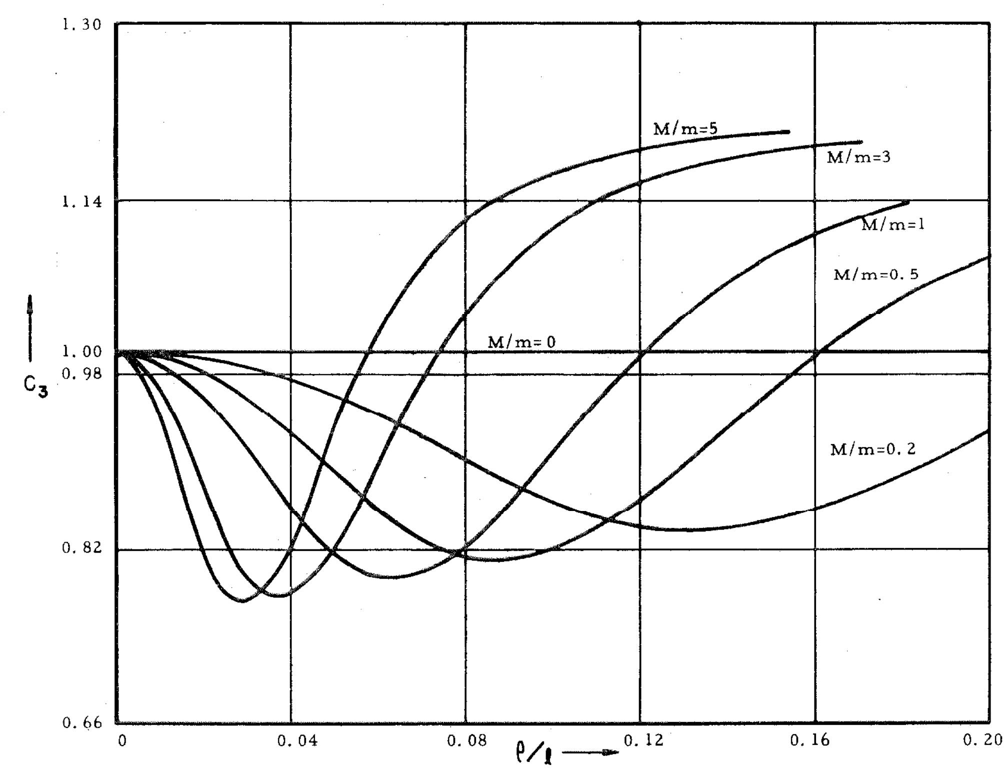
C-3-a. HINGED-HINGED, $\ell_1 = \ell/3$ - THIRD MODE - FREQUENCY ROOT



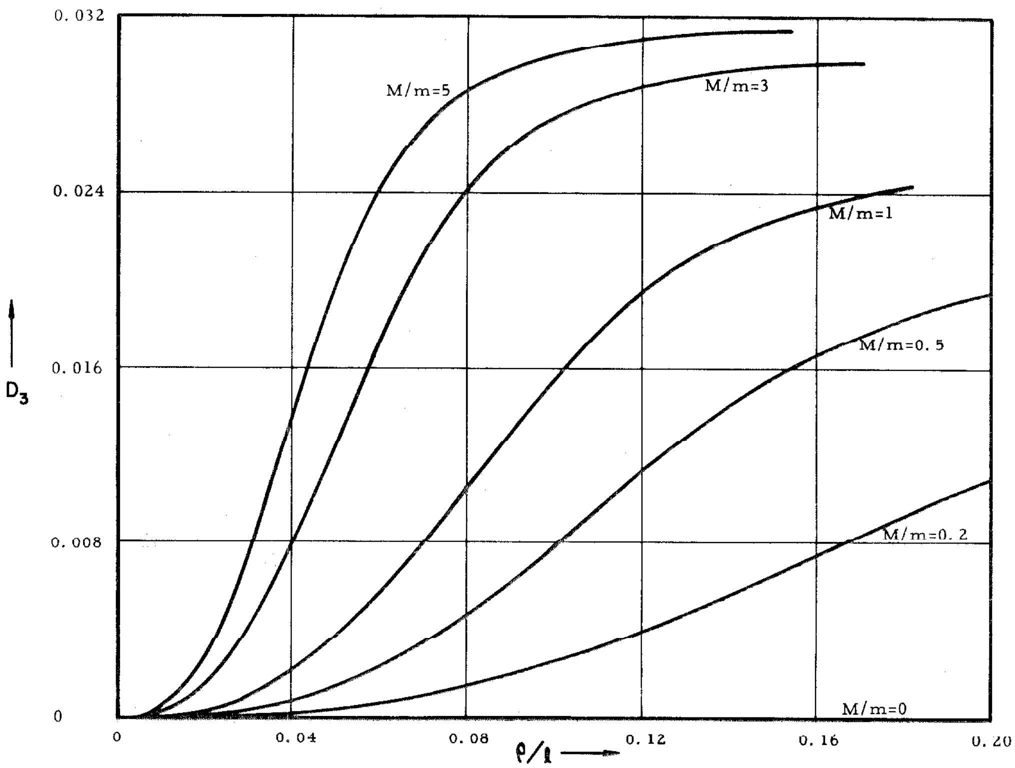
C-3-b(1). HINGED-HINGED, $\ell_1 = \ell/3$ - THIRD MODE - MODE SHAPE FACTOR



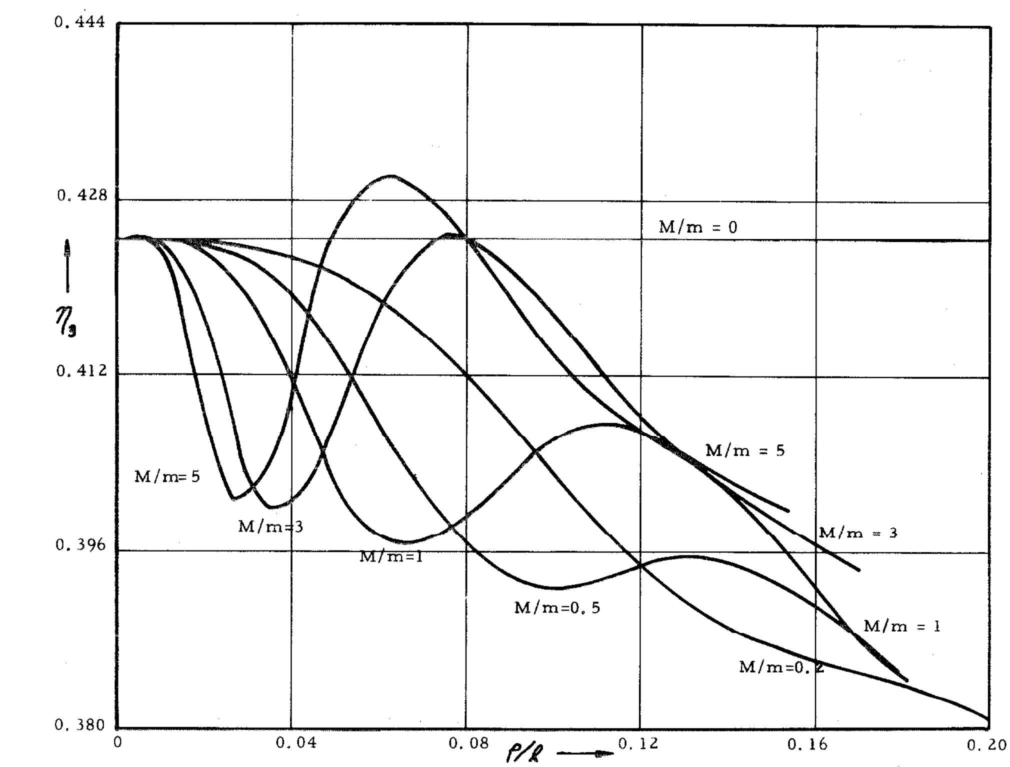
C-3-b(2). HINGED-HINGED, $l_1 = l/3$ - THIRD MODE - MODE SHAPE FACTOR



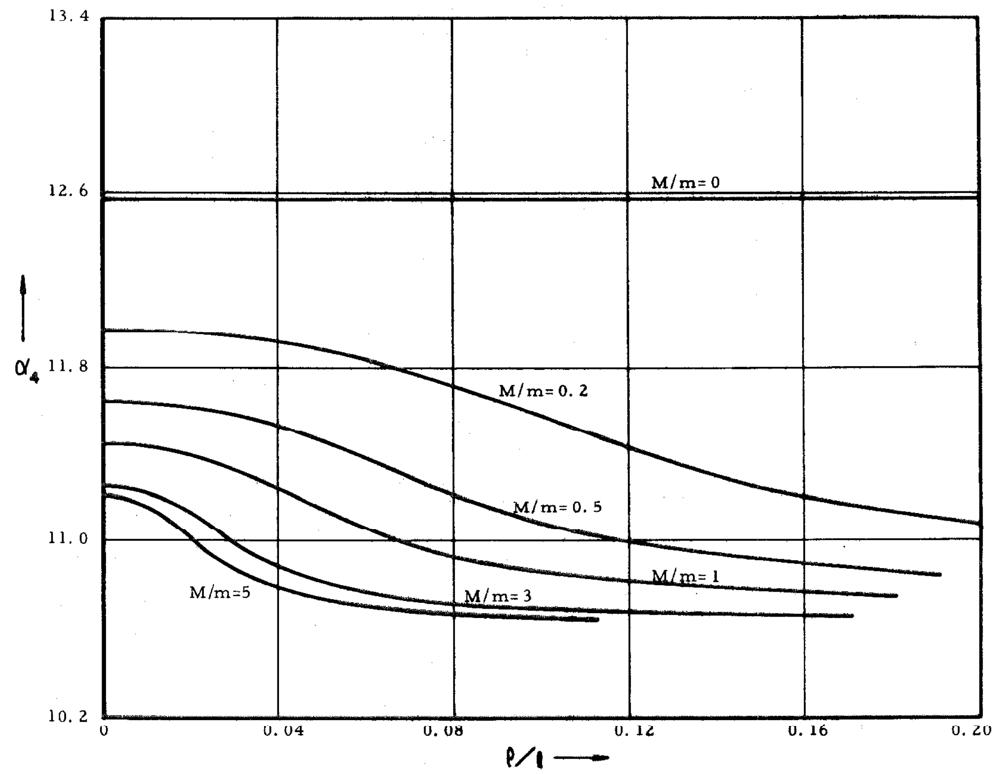
C-3-b(3). HINGED-HINGED, $l_1 = l/3$ - THIRD MODE - MODE SHAPE FACTOR



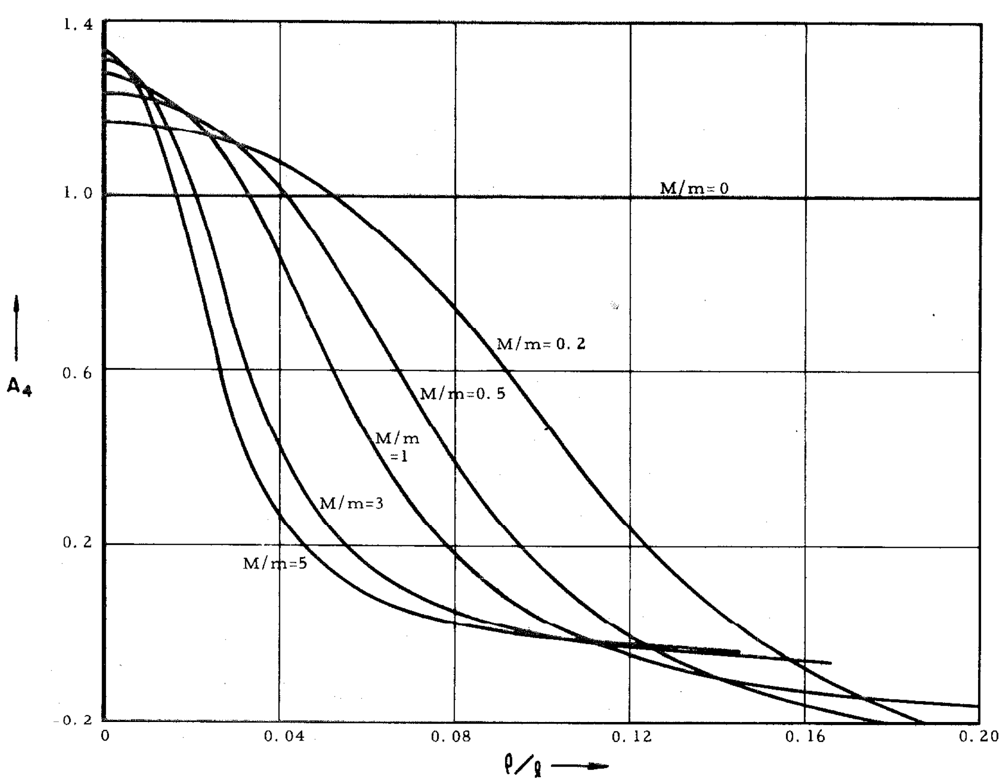
C-3-b(4). HINGED-HINGED, $\ell_1 = \ell/3$ - THIRD MODE - MODE SHAPE FACTOR



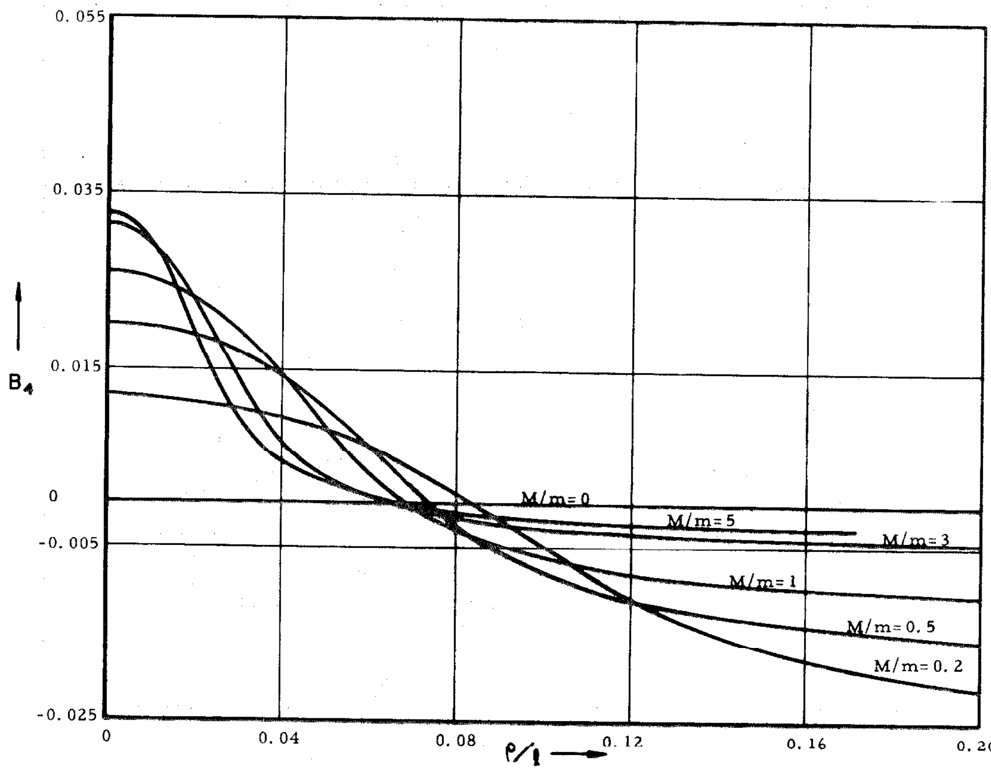
C-3-c. HINGED-HINGED, $\ell_1 = \ell/3$ - THIRD MODE - MODE PARTICIPATION FACTOR



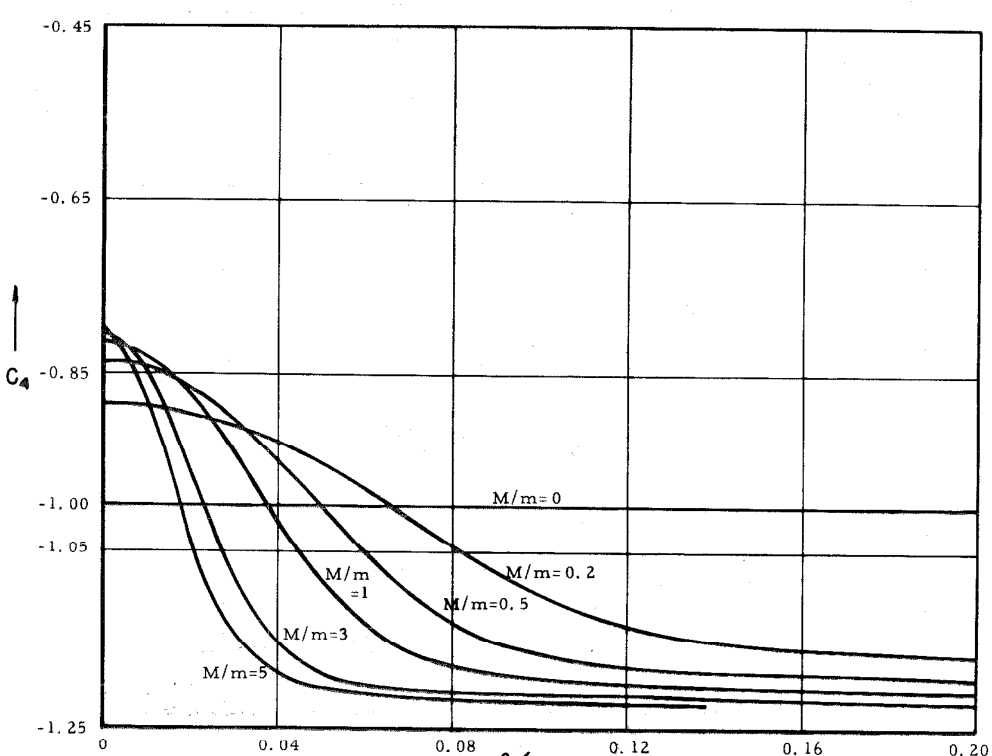
C-4-a. HINGED-HINGED, $l_1 = l/3$ - FOURTH MODE - FREQUENCY ROOT



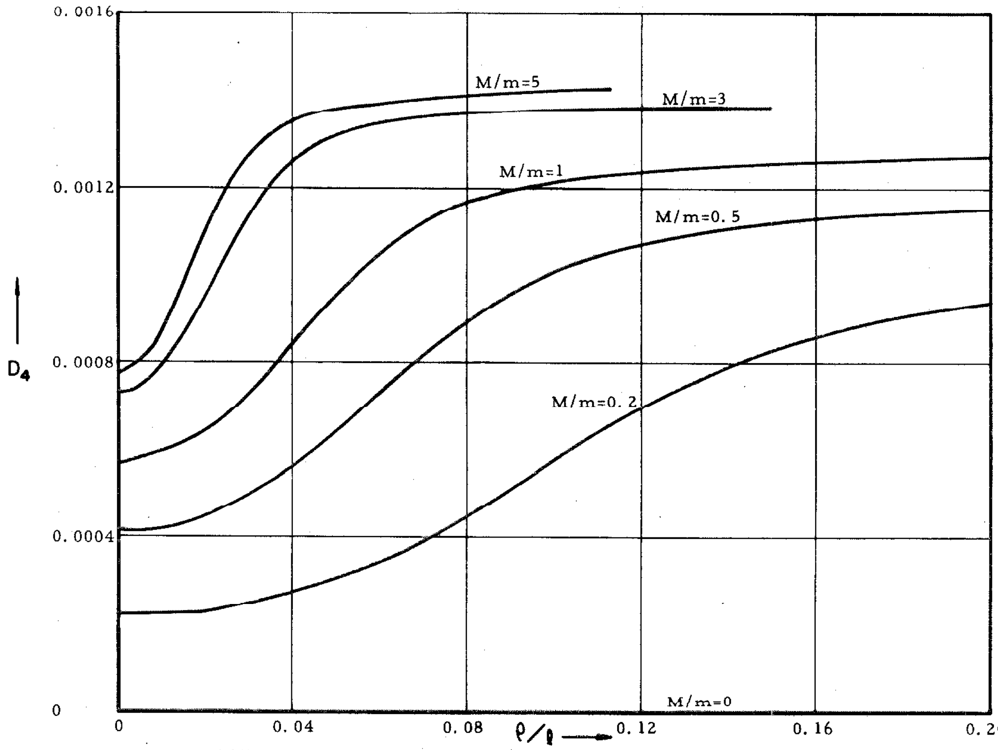
C-4-b(1). HINGED-HINGED, $l_1 = l/3$ - FOURTH MODE - MODE SHAPE FACTOR



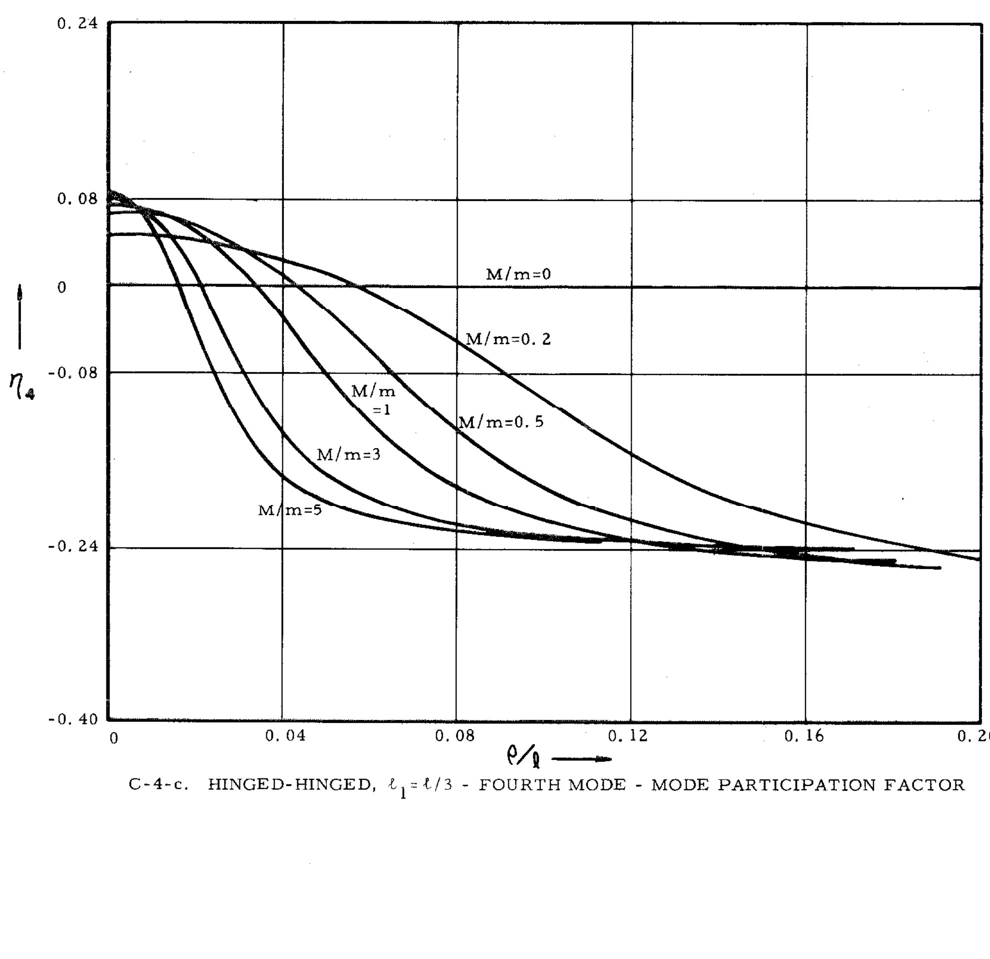
C-4-b(2). HINGED-HINGED, $\ell_1=\ell/3$ - FOURTH MODE - MODE SHAPE FACTOR



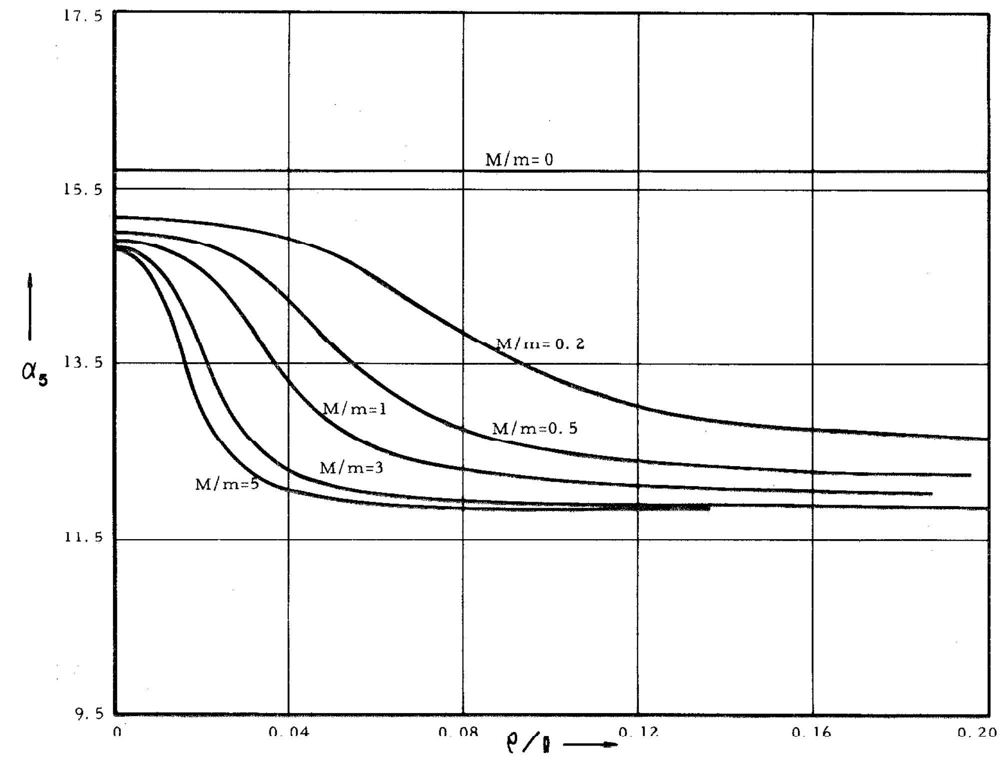
C-4-b(3). HINGED-HINGED, $\ell_1=\ell/3$ - FOURTH MODE - MODE SHAPE FACTOR



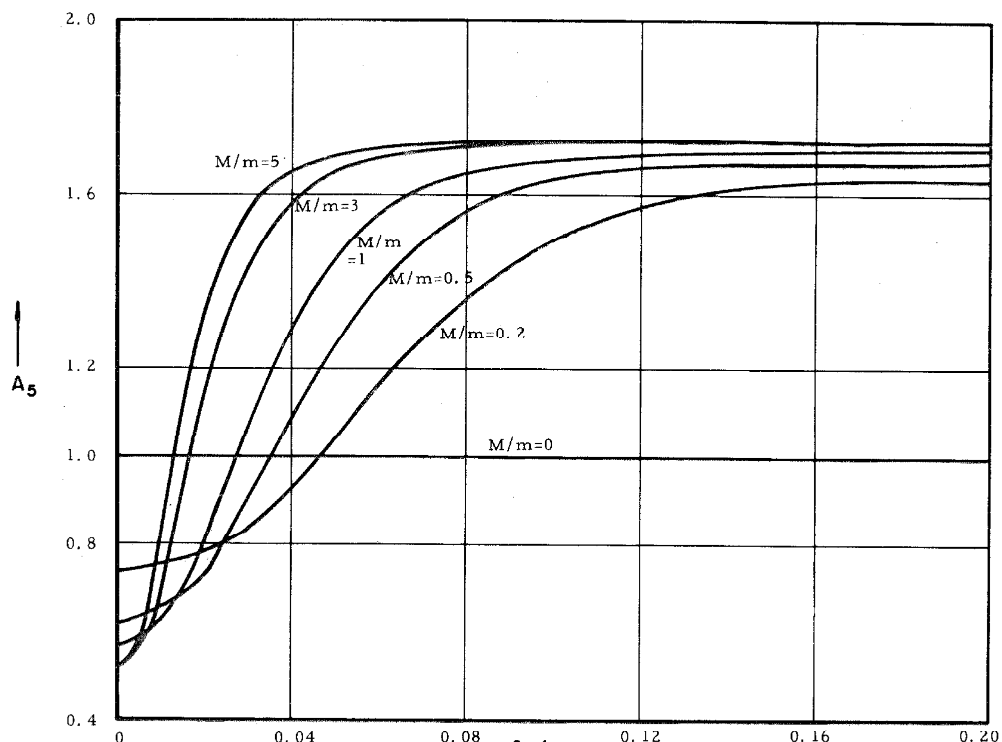
C-4-b(4). HINGED-HINGED, $\ell_1 = \ell/3$ - FOURTH MODE - MODE SHAPE FACTOR



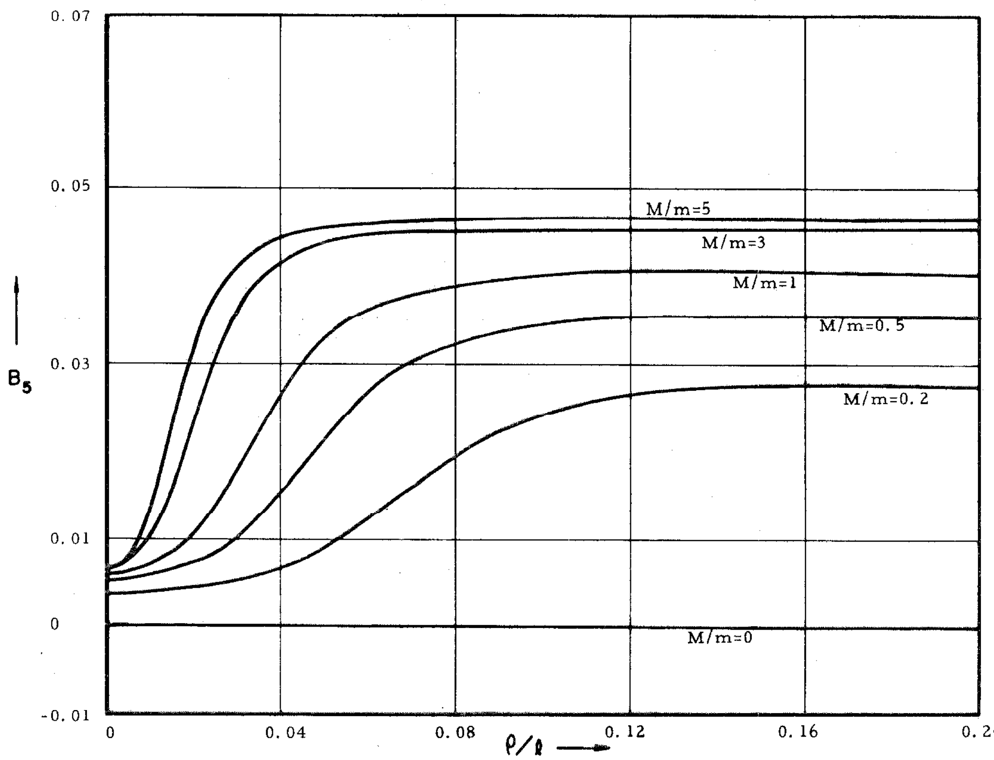
C-4-c. HINGED-HINGED, $\ell_1 = \ell/3$ - FOURTH MODE - MODE PARTICIPATION FACTOR



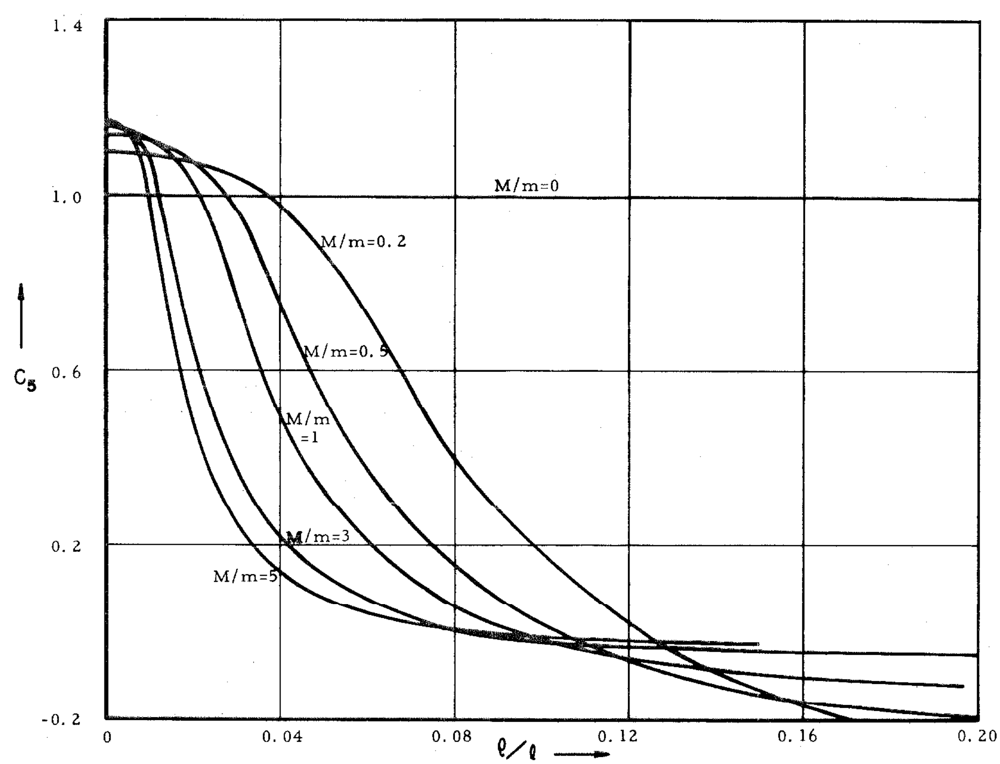
C-5-a. HINGED-HINGED, $\ell_1 = \ell/3$ - FIFTH MODE - FREQUENCY ROOT



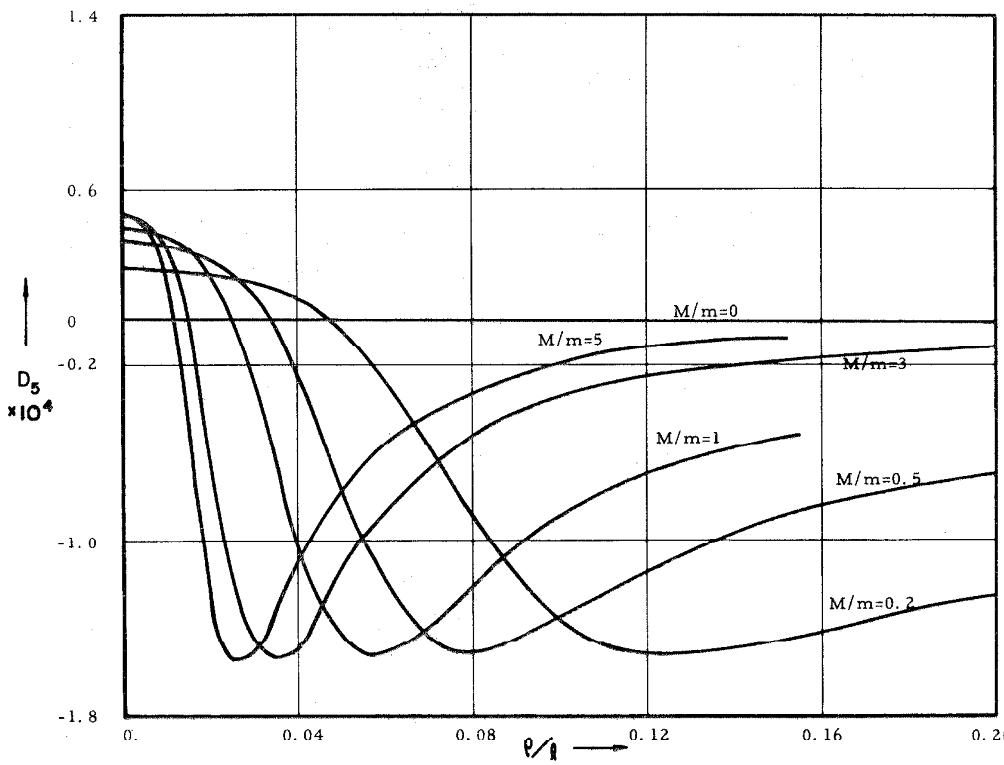
C-5-b(1). HINGED-HINGED, $\ell_1 = \ell/3$ - FIFTH MODE - MODE SHAPE FACTOR



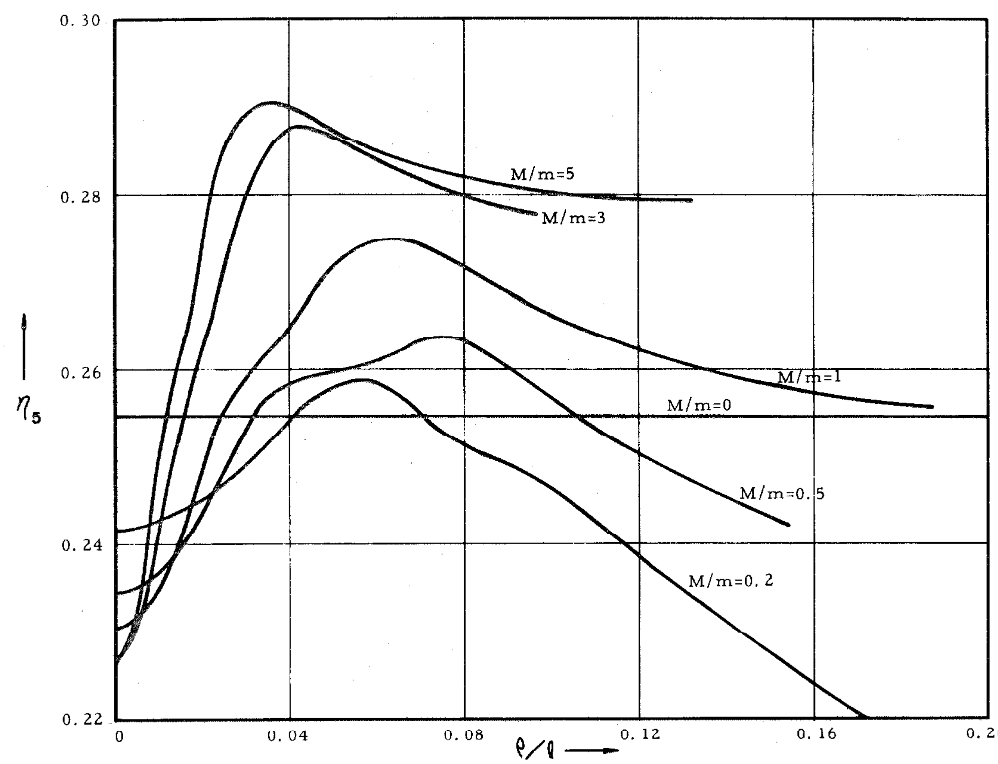
C-5-b(2). HINGED-HINGED, $\ell_1 = \ell/3$ - FIFTH MODE - MODE SHAPE FACTOR



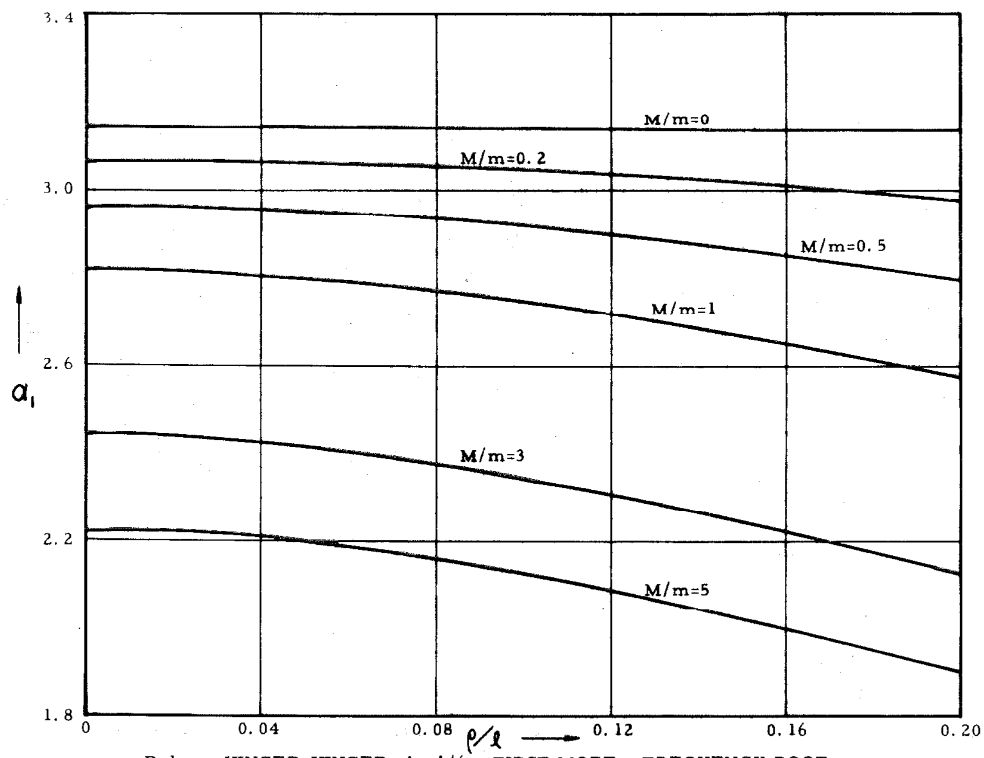
C-5-b(3). HINGED-HINGED, $\ell_1 = \ell/3$ - FIFTH MODE - MODE SHAPE FACTOR



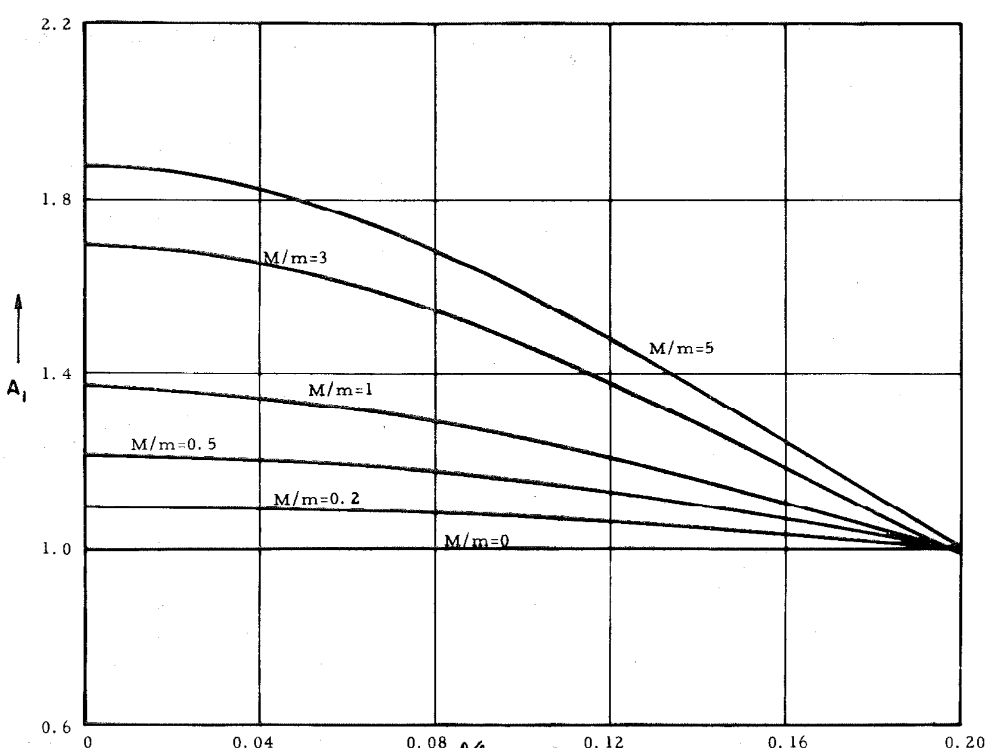
C-5-b(4). HINGED-HINGED, $l_1 = l/3$ - FIFTH MODE - MODE SHAPE FACTOR



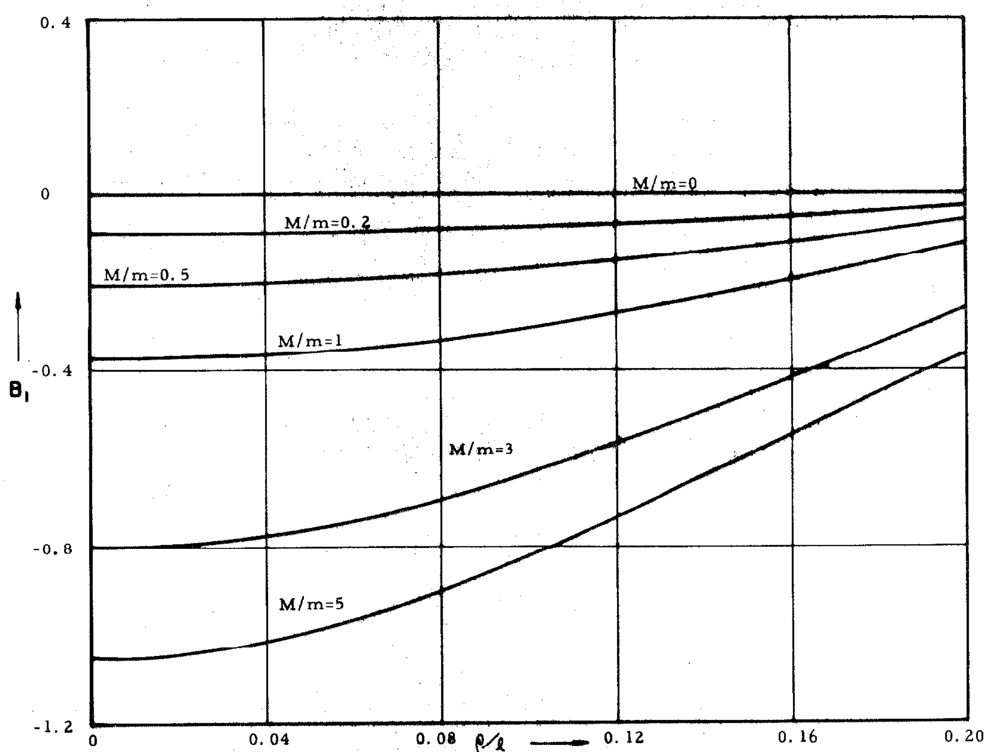
C-5-c. HINGED-HINGED, $l_1 = l/3$ - FIFTH MODE - MODE PARTICIPATION FACTOR



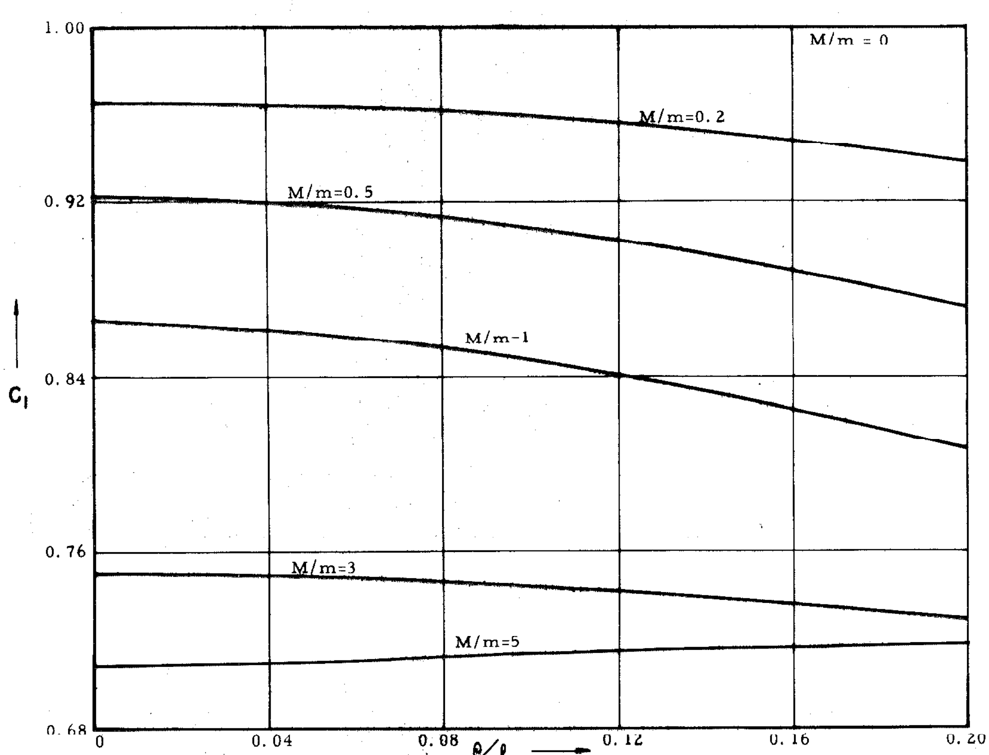
D-1-a. HINGED-HINGED, $\ell_1 = \ell/6$ - FIRST MODE - FREQUENCY ROOT



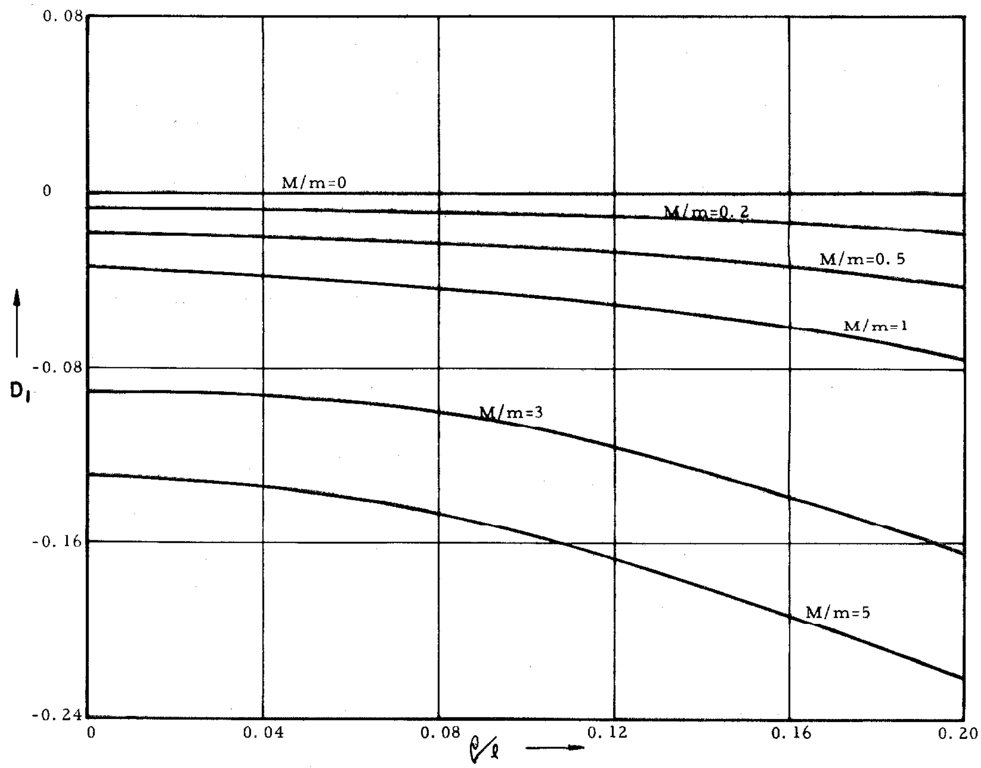
D-1-b(1). HINGED-HINGED, $\ell_1 = \ell/6$ - FIRST MODE - MODE SHAPE FACTOR



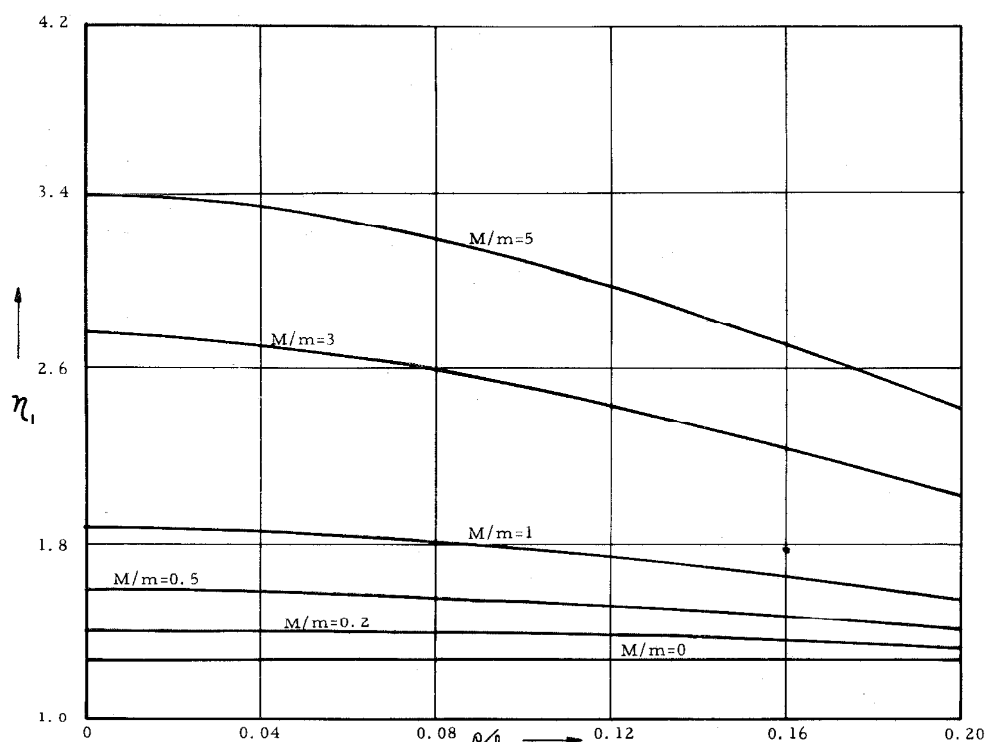
D-1-b(2). HINGED-HINGED, $\ell_1 = \ell/6$ - FIRST MODE - MODE SHAPE FACTOR



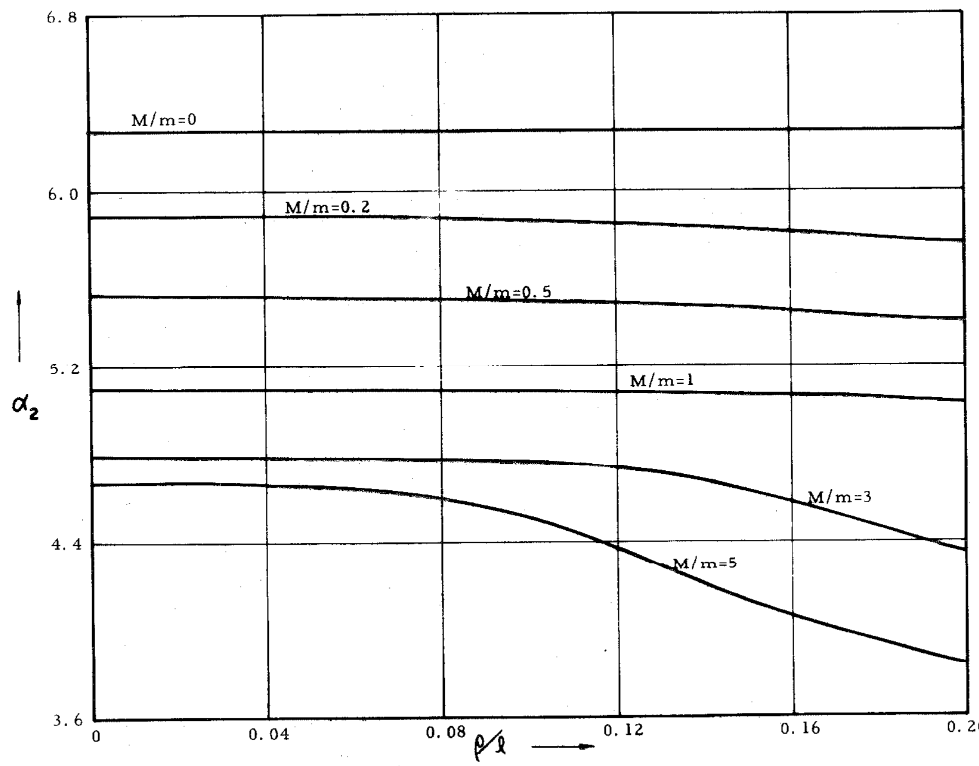
D-1-b(3). HINGED-HINGED, $\ell_1 = \ell/6$ - FIRST MODE - MODE SHAPE FACTOR



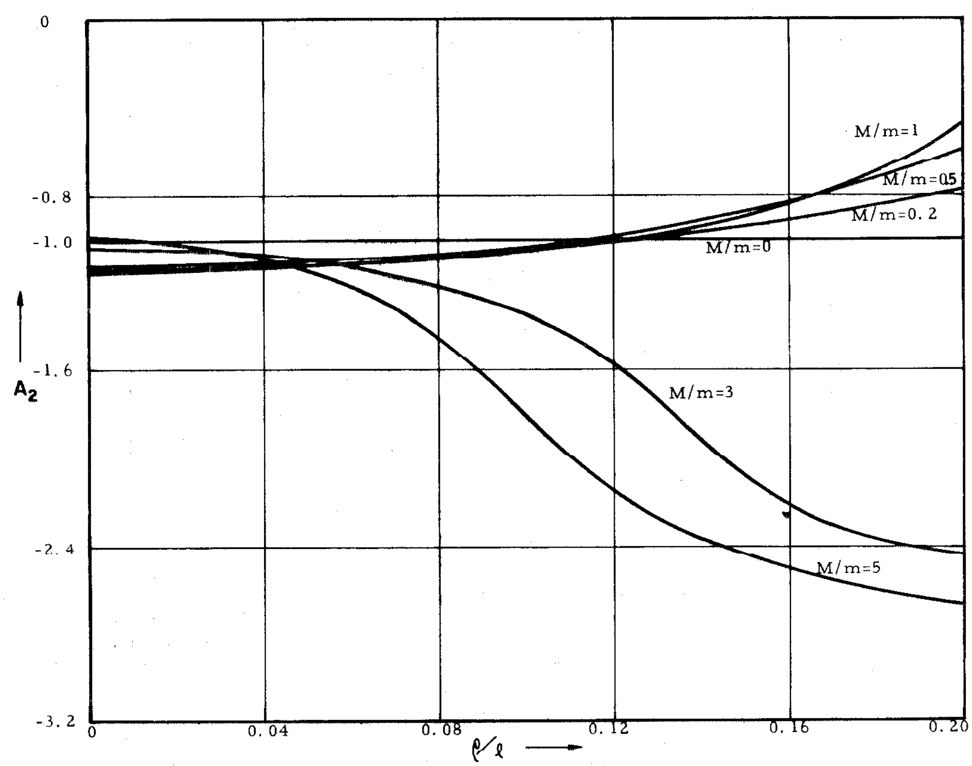
D-1-b(4). HINGED-HINGED, $\ell_1=\ell/6$ - FIRST MODE - MODE SHAPE FACTOR



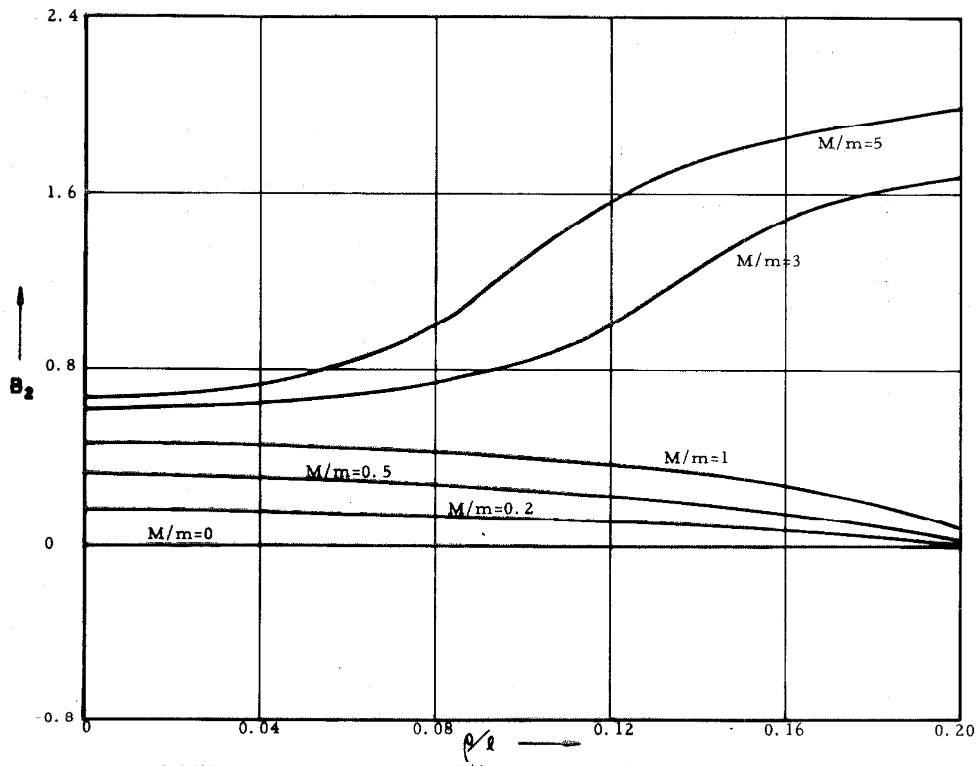
D-1-c. HINGED-HINGED, $\ell_1=\ell/6$ - FIRST MODE - MODE PARTICIPATION FACTOR



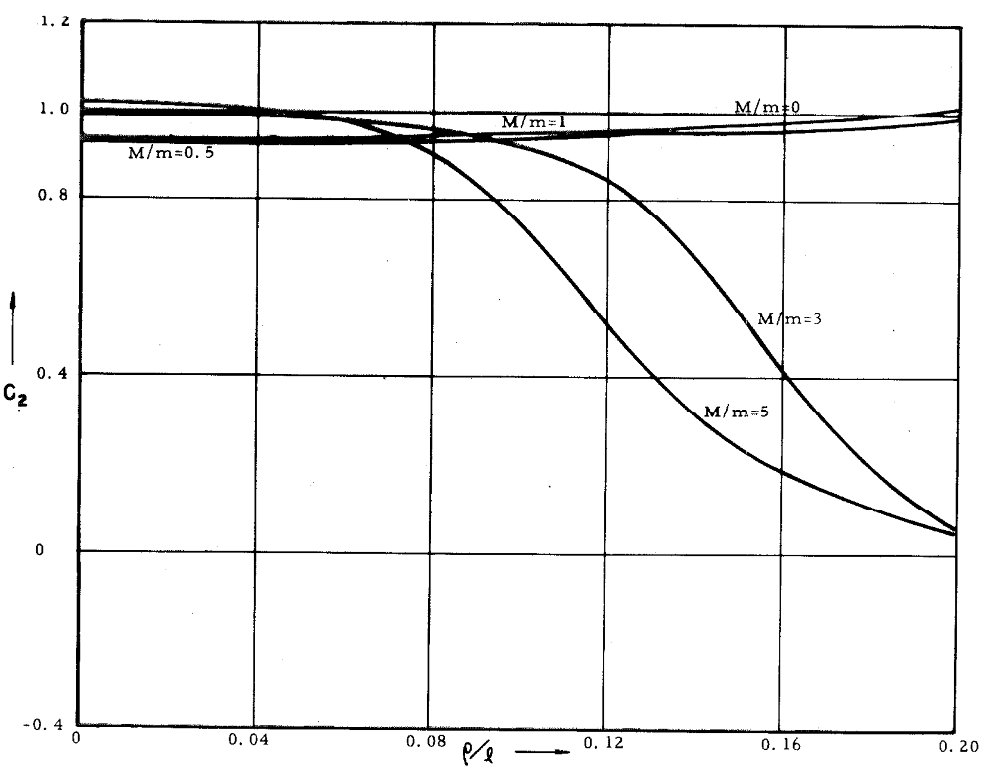
D-2-a. HINGED-HINGED, $\ell_1 = \ell/6$ - SECOND MODE - FREQUENCY ROOT



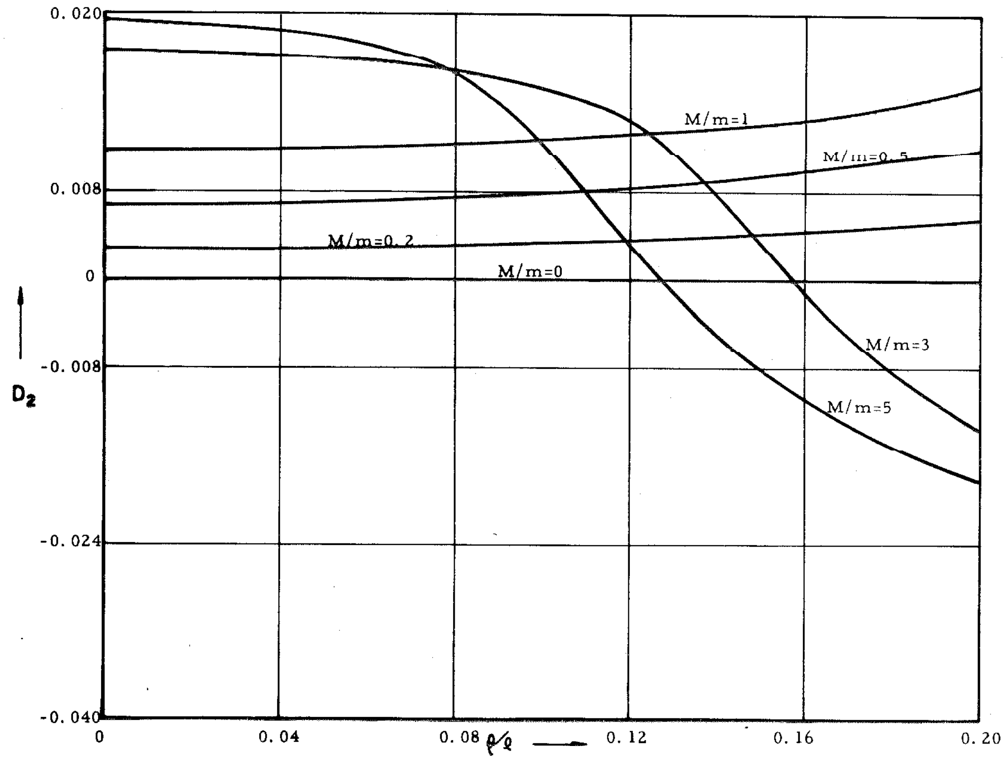
D-2-b(1). HINGED-HINGED, $\ell_1 = \ell/6$ - SECOND MODE - MODE SHAPE FACTOR



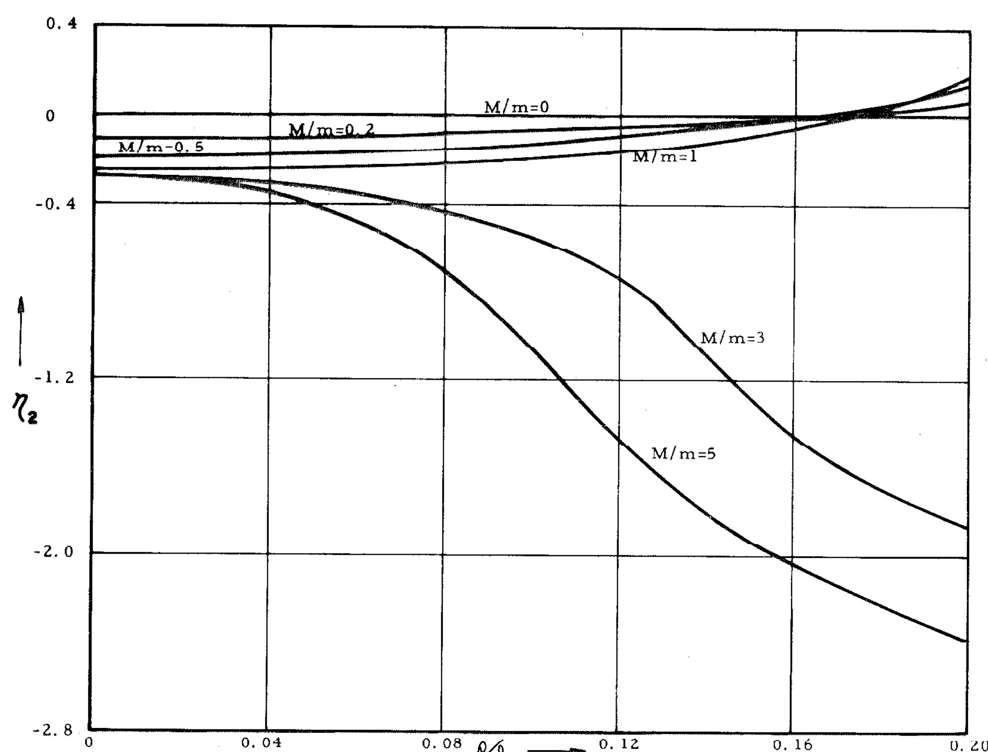
D-2-b(2). HINGED-HINGED, $\ell_1 = \ell/6$ - FIRST MODE - MODE SHAPE FACTOR



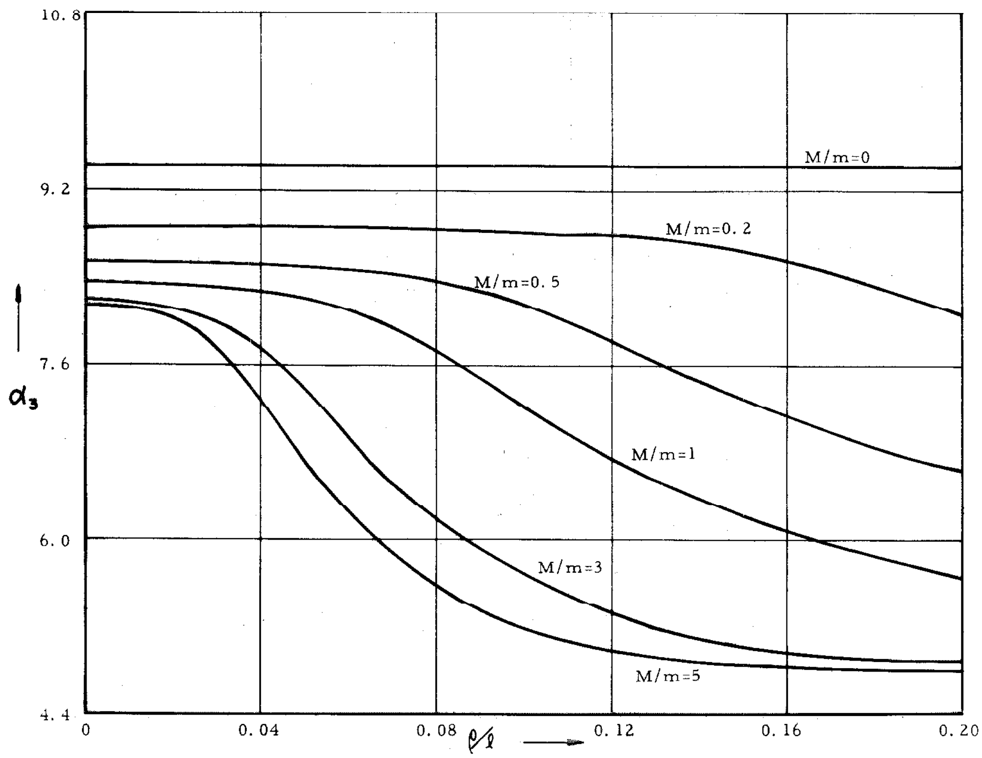
D-2-b(3). HINGED-HINGED, $\ell_1 = \ell/6$ - SECOND MODE - MODE SHAPE FACTOR



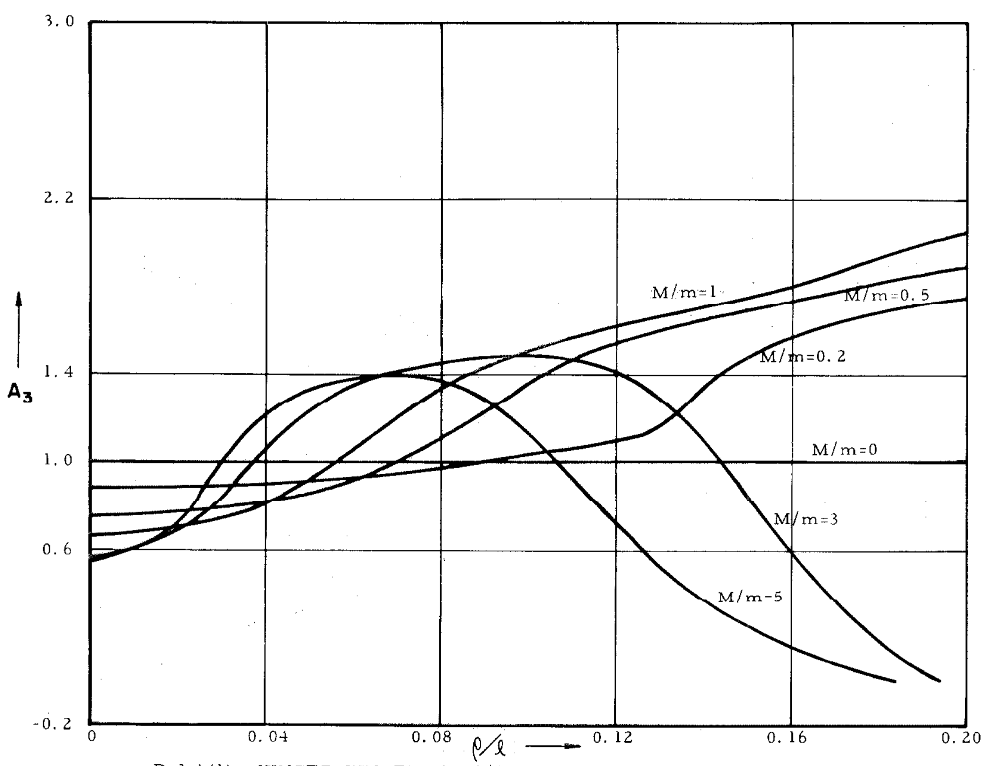
D-2-b(4). HINGED-HINGED, $\ell_1 = \ell/6$ - SECOND MODE - MODE SHAPE FACTOR



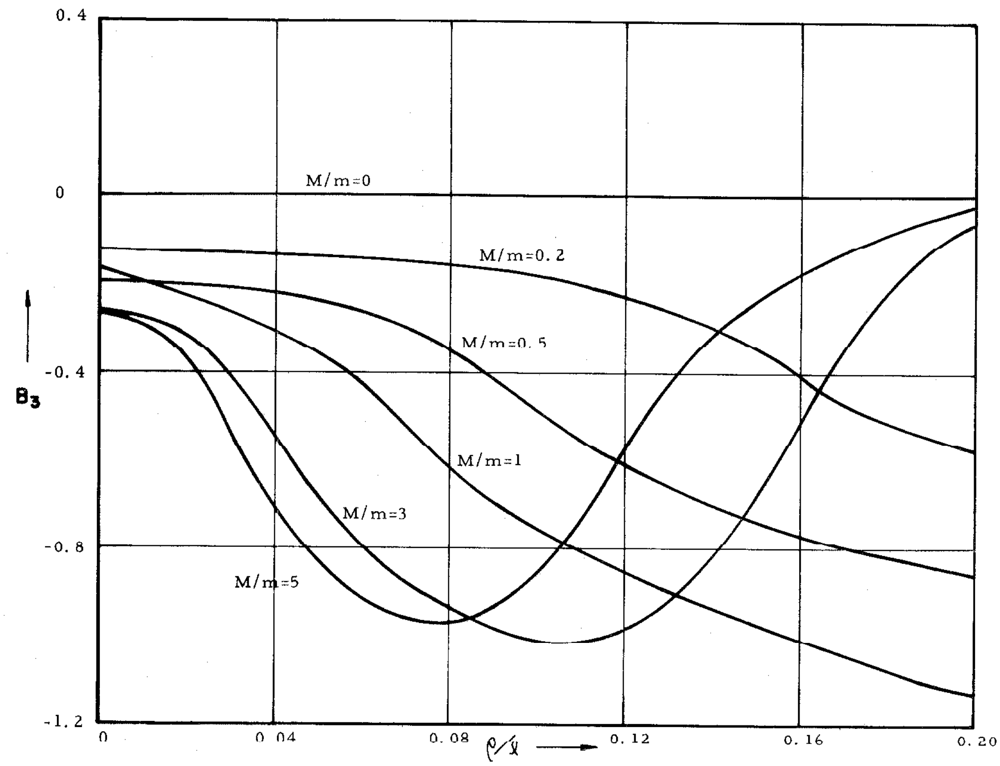
D-2-c. HINGED-HINGED, $\ell_1 = \ell/6$ - SECOND MODE - MODE PARTICIPATION FACTOR



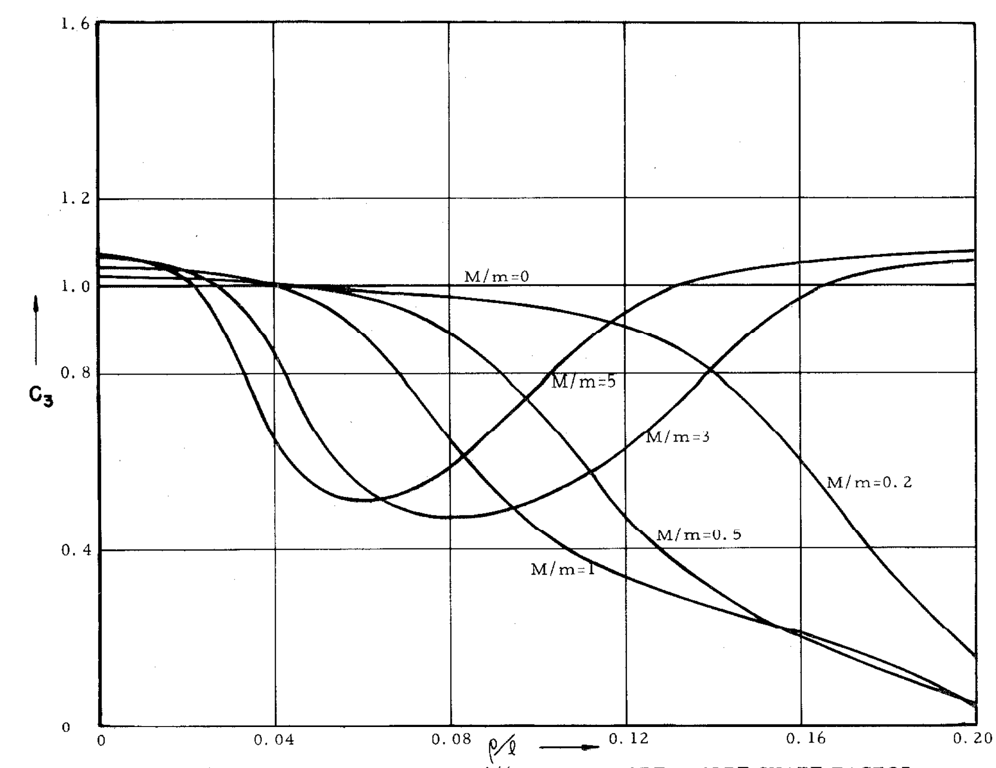
D-3-a. HINGED-HINGED, $\ell_1 = \ell/6$ - THIRD MODE - FREQUENCY ROOT



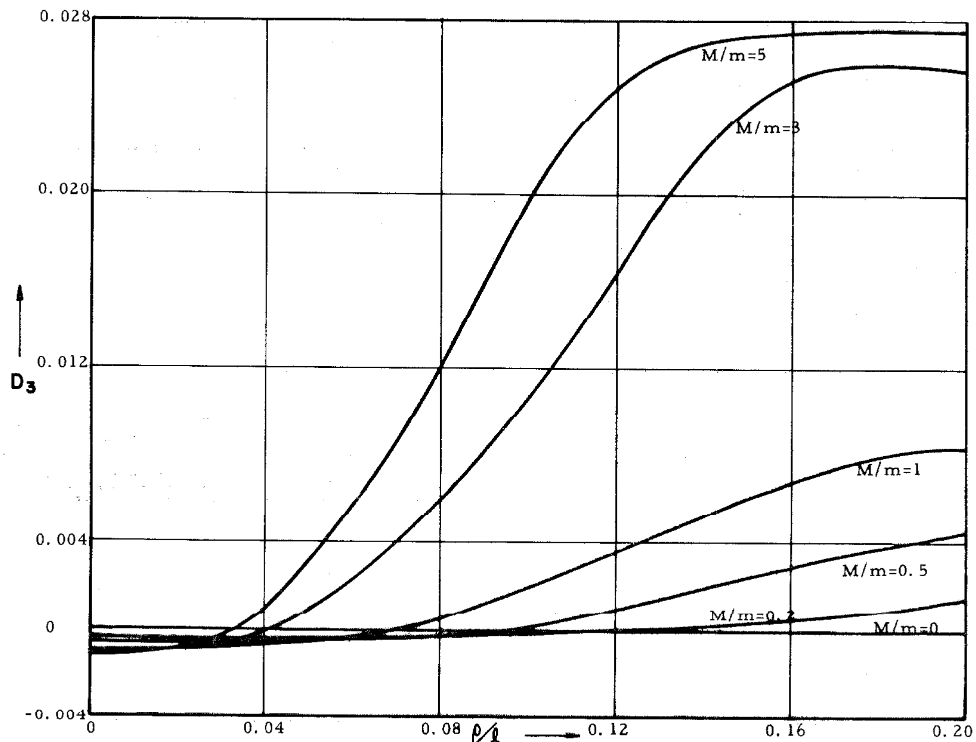
D-3-b(1). HINGED-HINGED, $\ell_1 = \ell/6$ - THIRD MODE - MODE SHAPE FACTOR



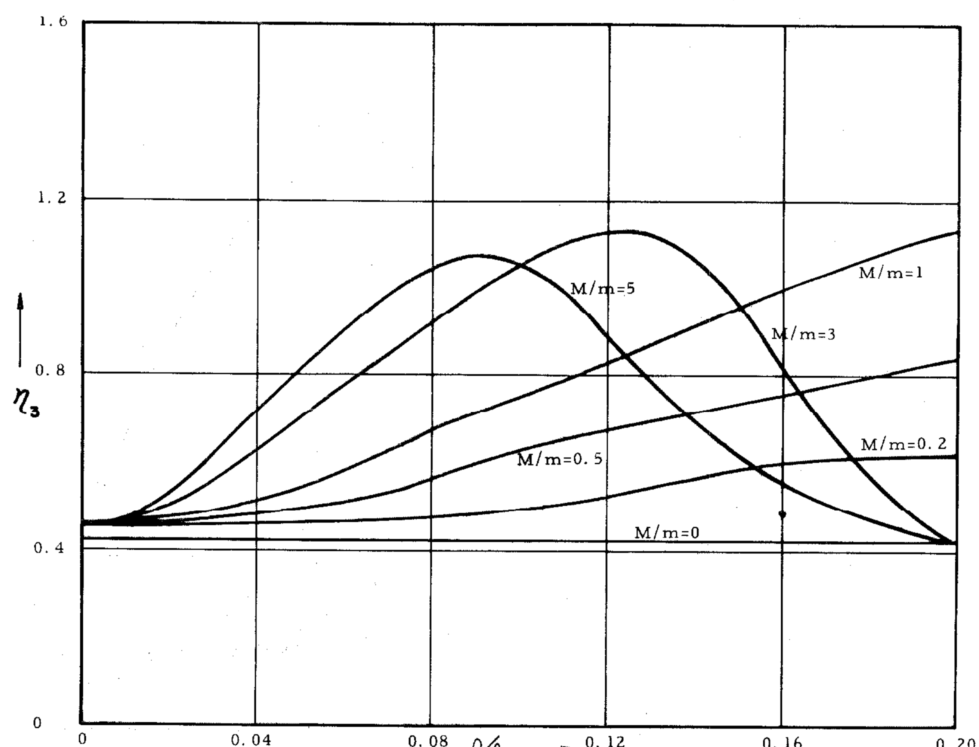
D-3-b(2). HINGED-HINGED, $\ell_1 = \ell/6$ - THIRD MODE - MODE SHAPE FACTOR



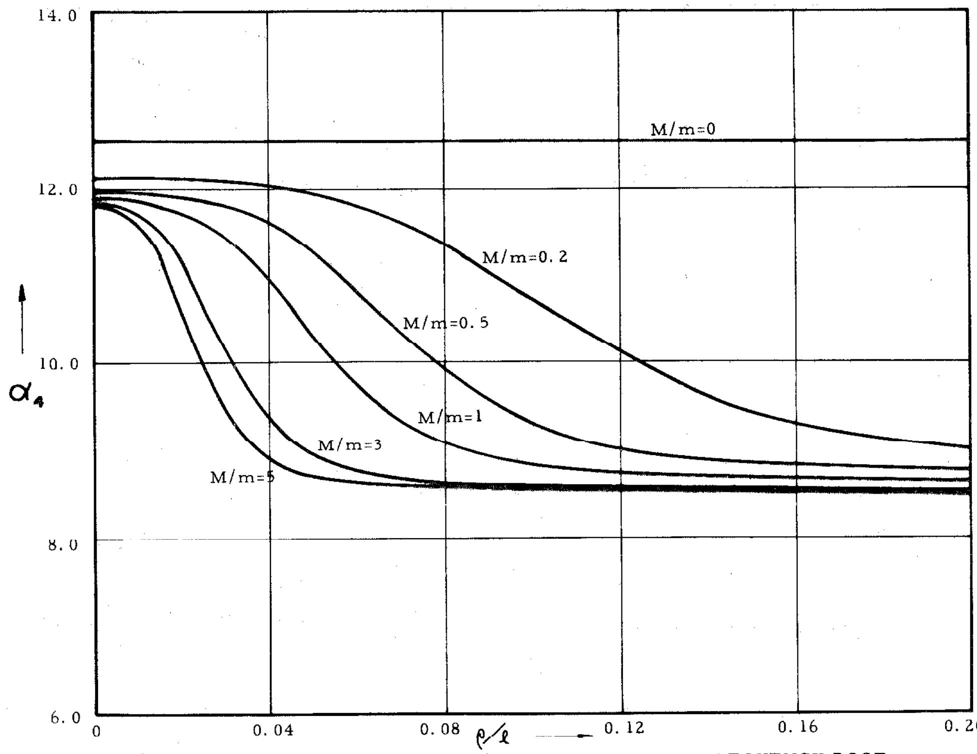
D-3-b(3). HINGED-HINGED, $\ell_1 = \ell/6$ - THIRD MODE - MODE SHAPE FACTOR



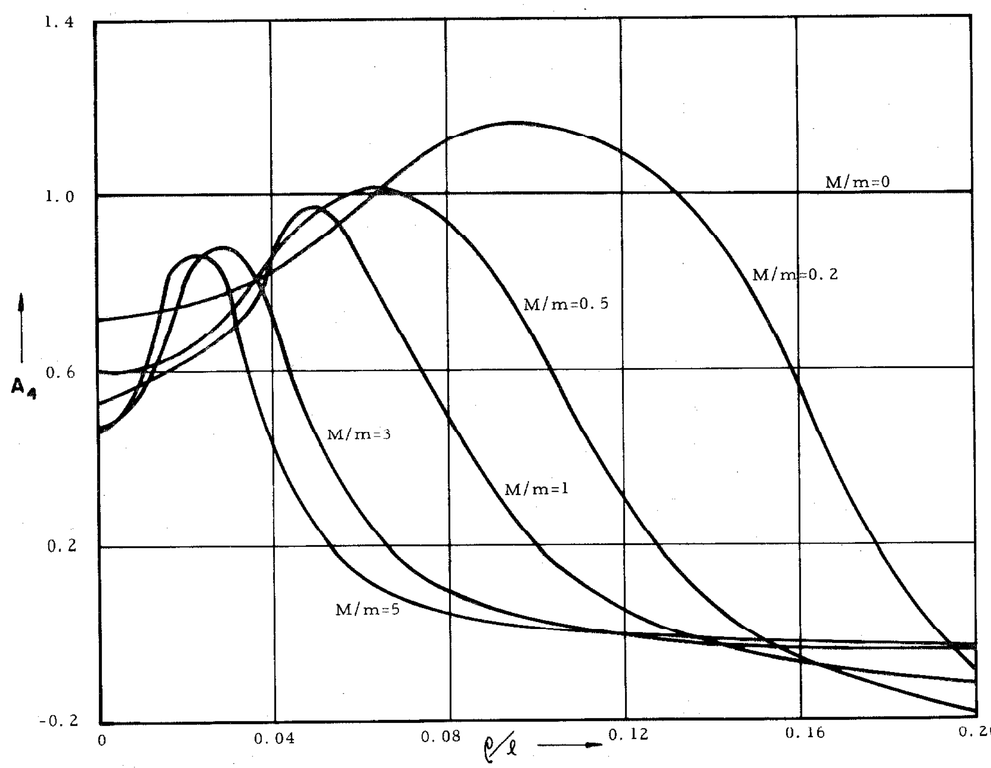
D-3-b(4). HINGED-HINGED, $\ell_1 = \ell/6$ - THIRD MODE - MODE SHAPE FACTOR



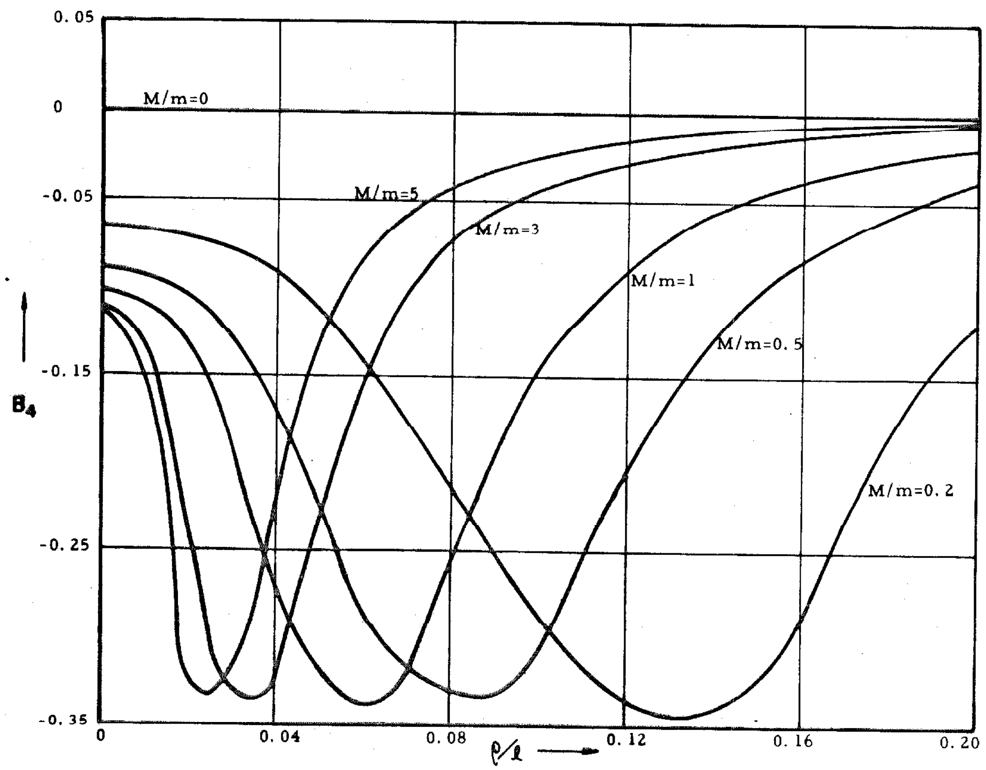
D-3-c. HINGED-HINGED, $\ell_1 = \ell/6$ - THIRD MODE - MODE PARTICIATION FACTOR



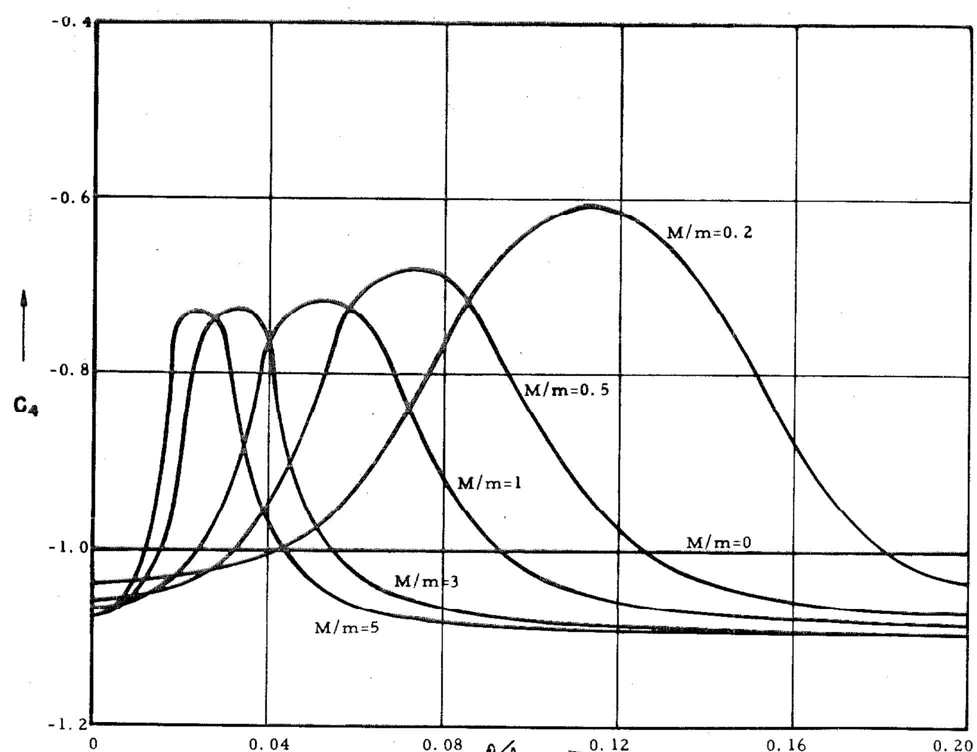
D-4-a. HINGED-HINGED, $\ell_1 = \ell/6$ - FOURTH MODE - FREQUENCY ROOT



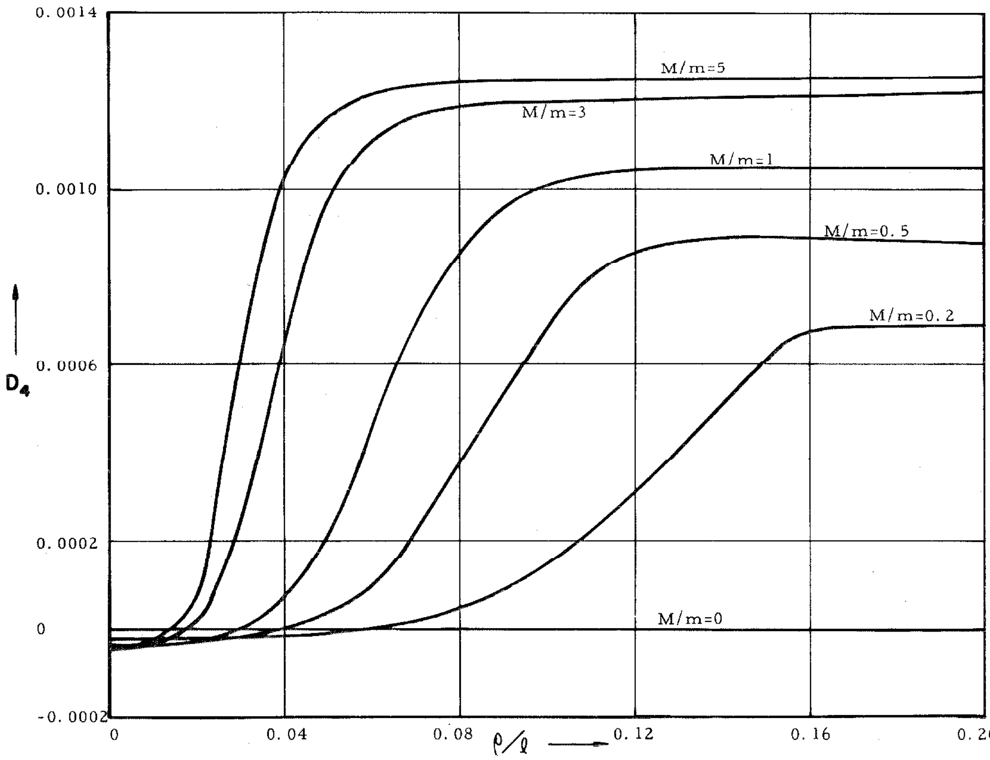
D-4-b(1). HINGED-HINGED, $\ell_1 = \ell/6$ - FOURTH MODE - MODE SHAPE FACTOR



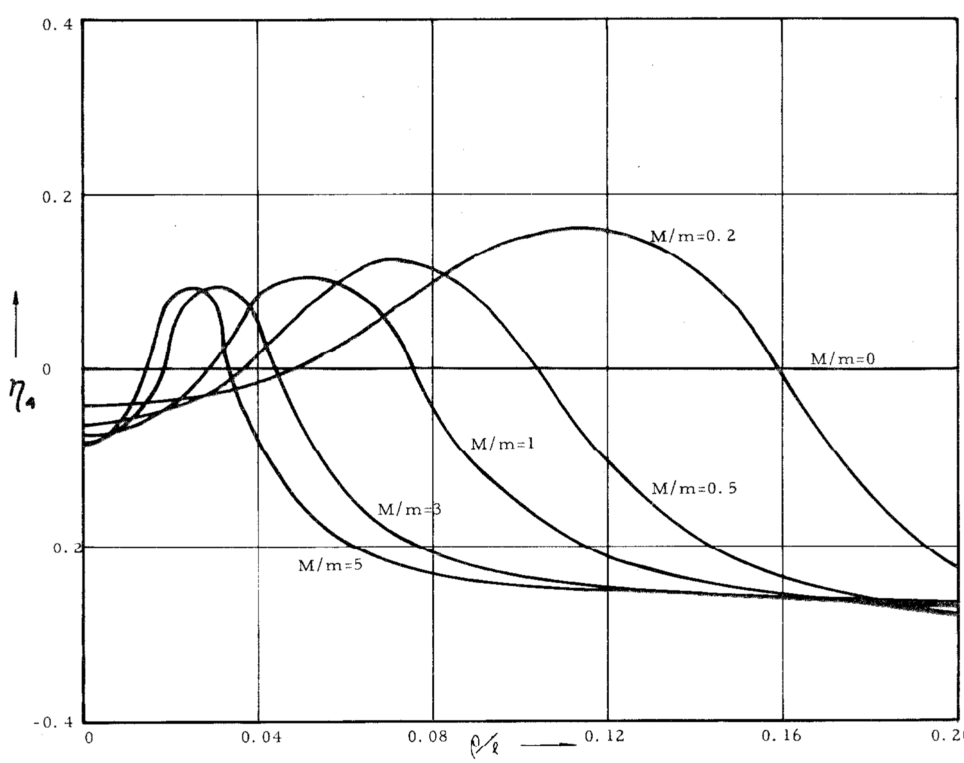
D-4-b(2). HINGED-HINGED, $\ell_1 = \ell/6$ - FOURTH MODE - MODE SHAPE FACTOR



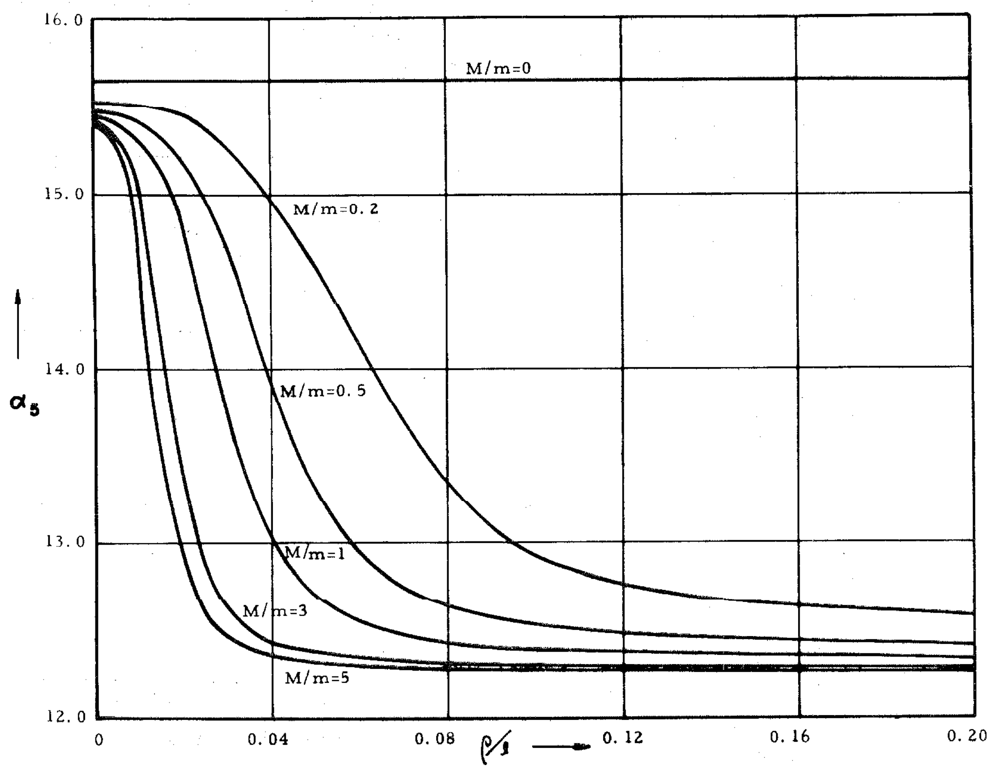
D-4-b(3). HINGED-HINGED, $\ell_1 = \ell/6$ - FOURTH MODE - MODE SHAPE FACTOR



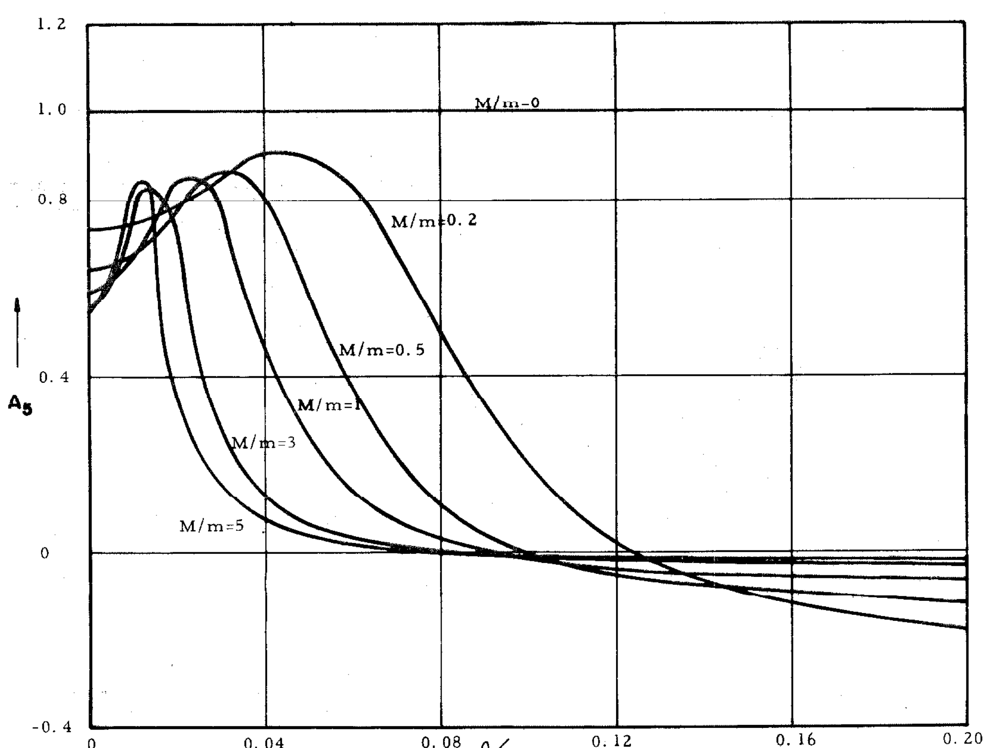
D-4-b(4). HINGED-HINGED, $\ell_1 = \ell/6$ - FOURTH MODE - MODE SHAPE FACTOR



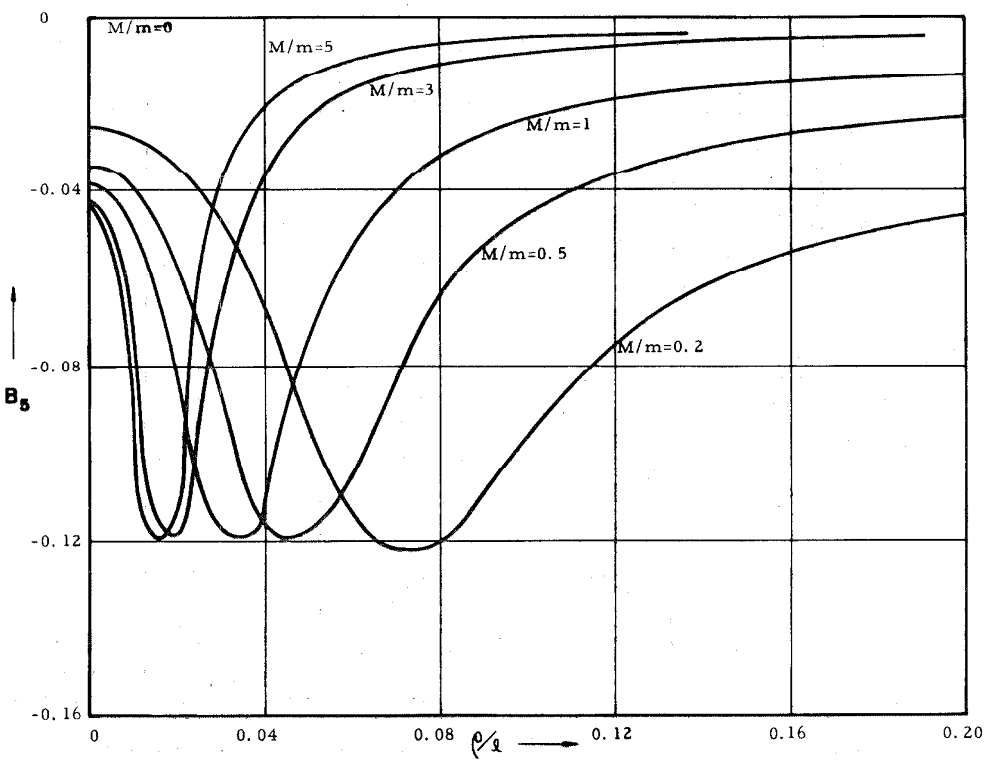
D-4-c. HINGED-HINGED, $\ell_1 = \ell/6$ - FOURTH MODE - MODE PARTICIPATION FACTOR



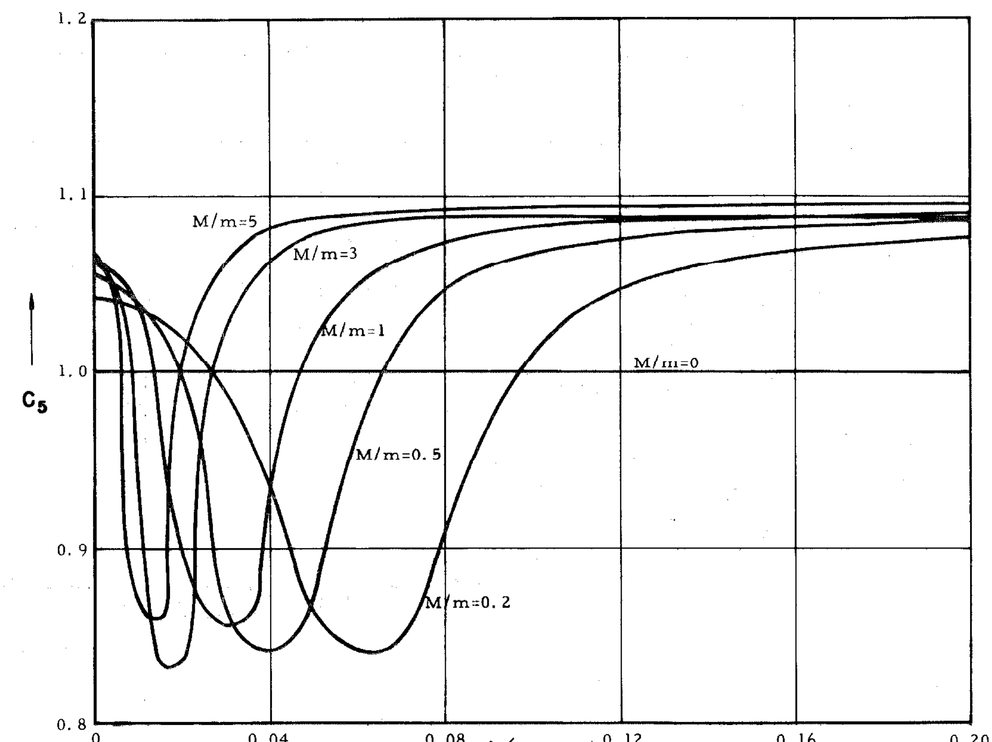
D-5-a. HINGED-HINGED, $\ell_1 = \ell/6$ - FIFTH MODE - FREQUENCY ROOT



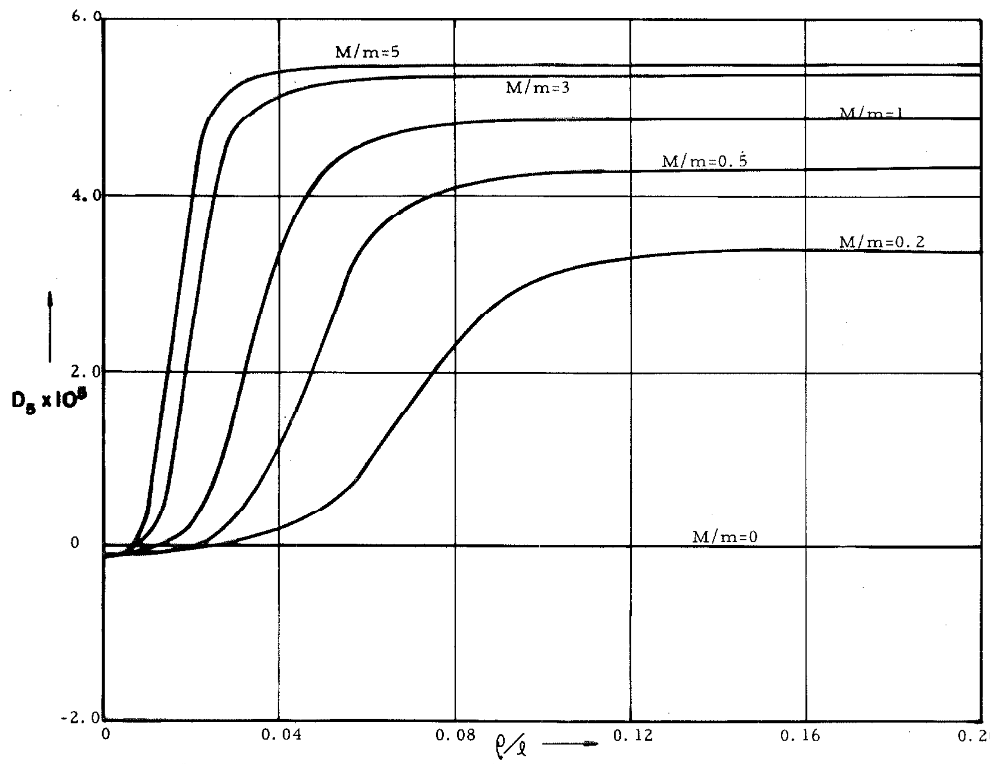
D-5-b(1). HINGED-HINGED, $\ell_1 = \ell/6$ - FIFTH MODE - MODE SHAPE FACTOR



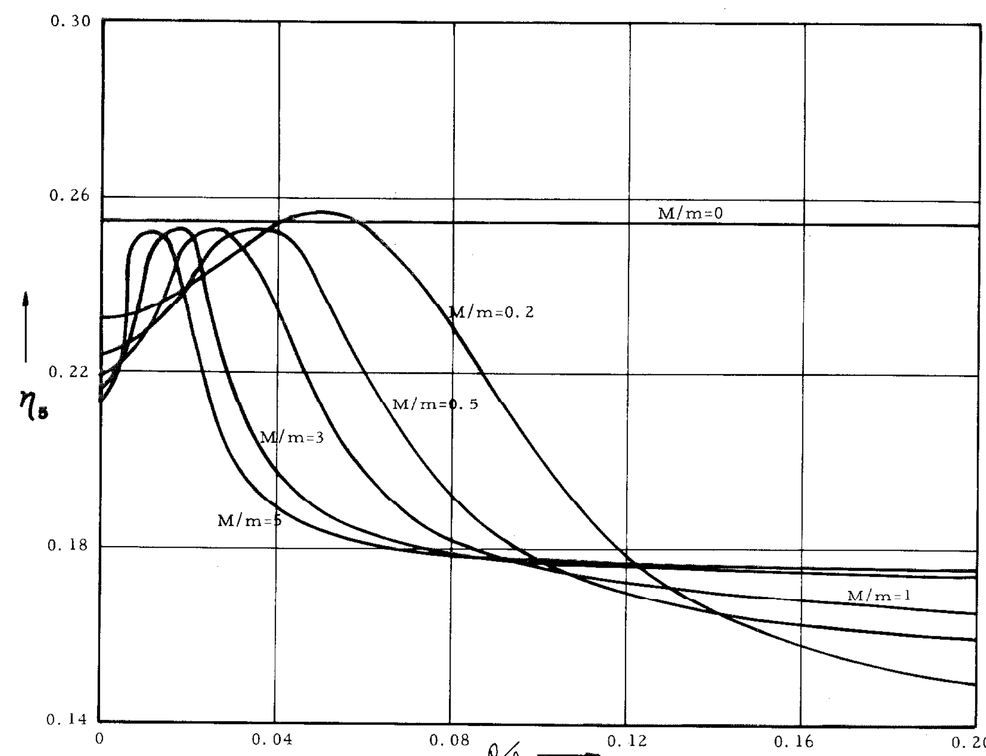
D-5-b(2). HINGED-HINGED, $\ell_1 = \ell/6$ - FIFTH MODE - MODE SHAPE FACTOR



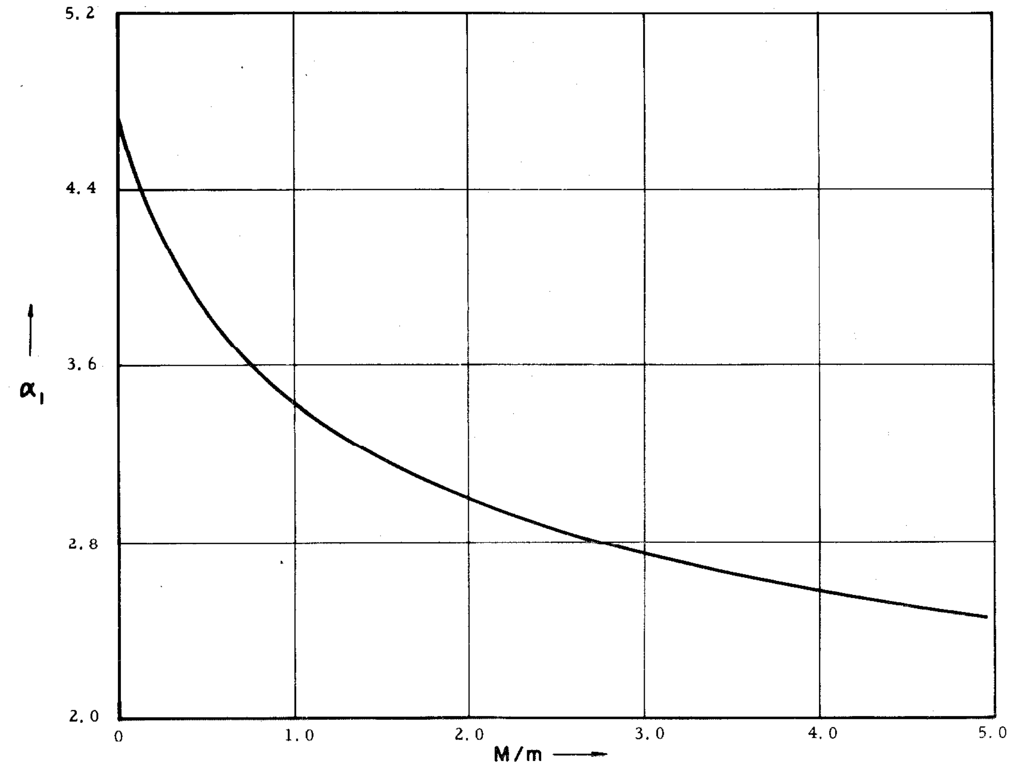
D-5-b(3). HINGED-HINGED, $\ell_1 = \ell/6$ - FIFTH MODE - MODE SHAPE FACTOR



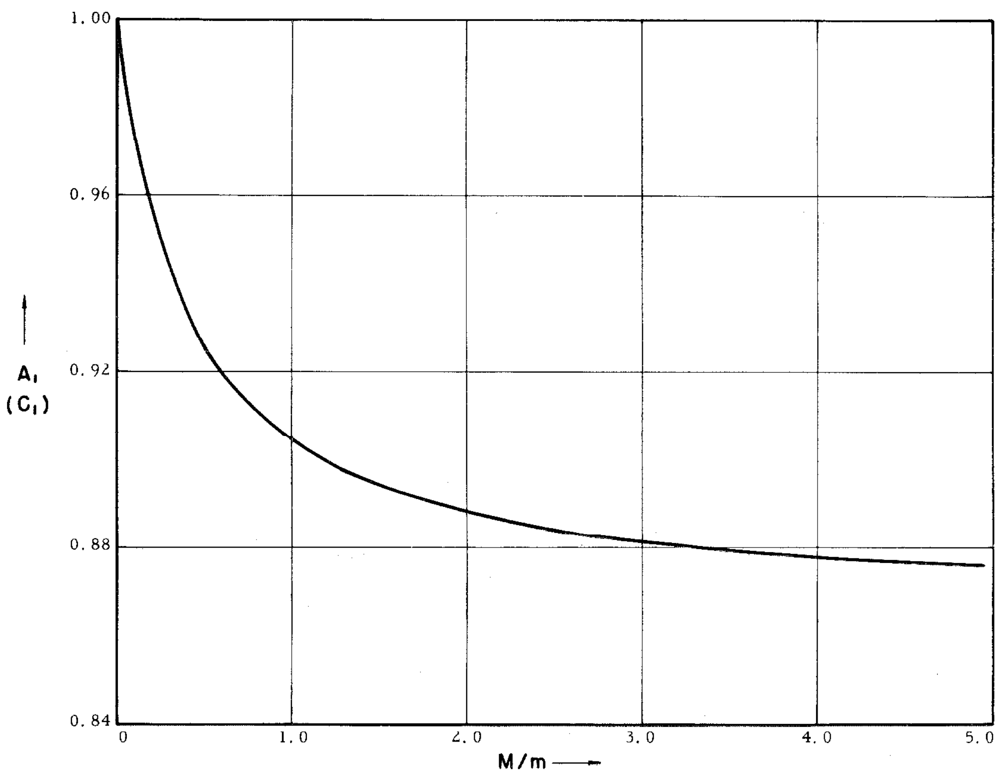
D-5-b(4). HINGED-HINGED, $\ell_1 = \ell/6$ - FIFTH MODE - MODE SHAPE FACTOR



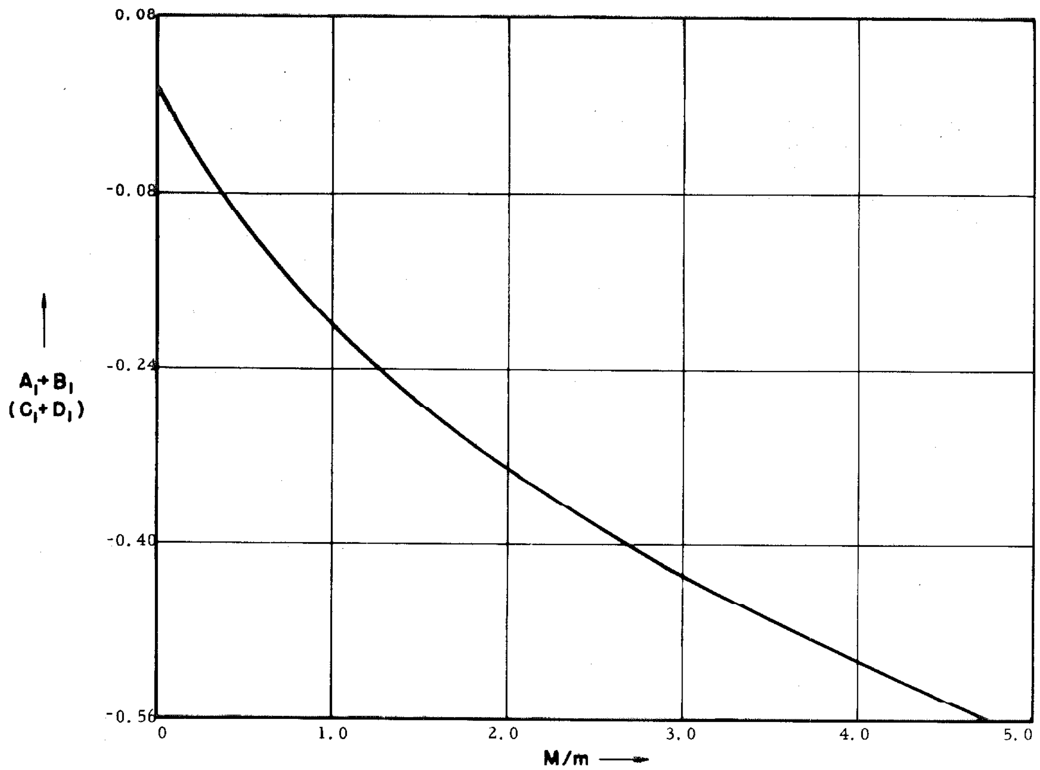
D-5-c. HINGED-HINGED, $\ell_1 = \ell/6$ - FIFTH MODE - MODE PARTICIPATION FACTOR



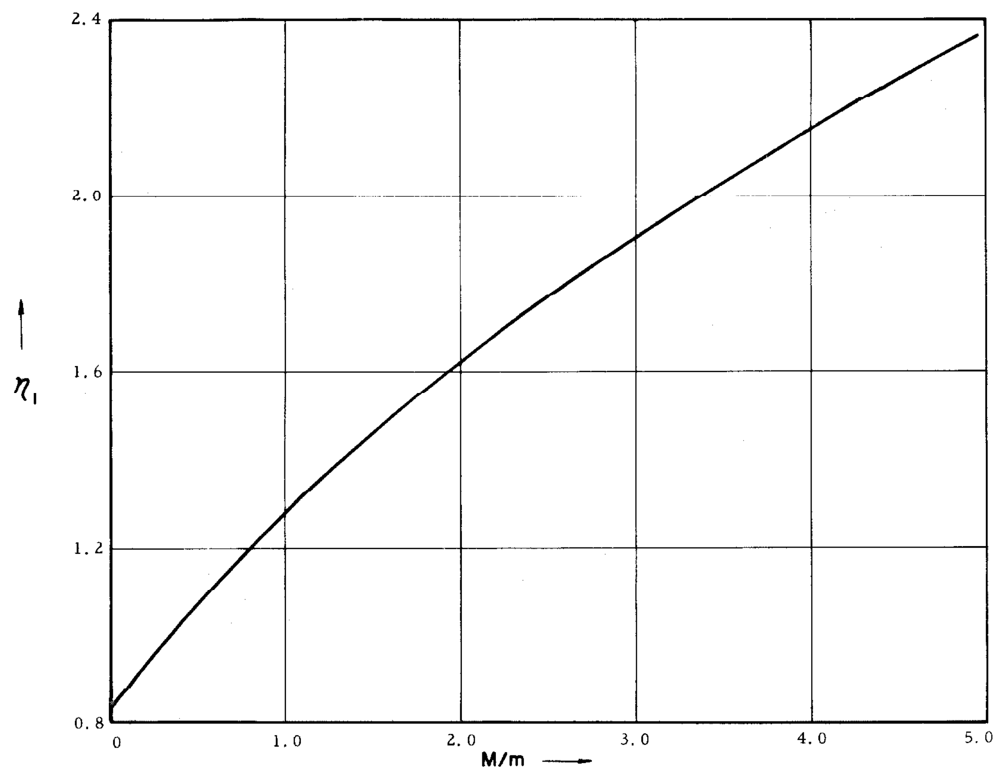
E-1-a. FIXED-FIXED, $\ell_1 = \ell/2$ - FIRST MODE - FREQUENCY ROOT



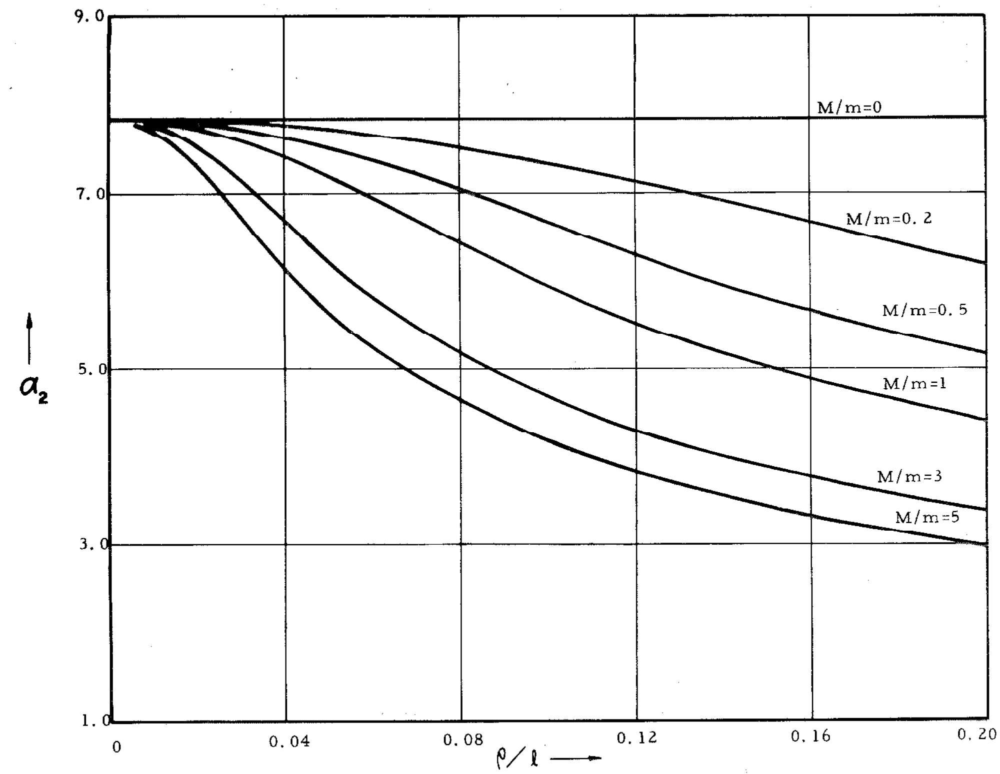
E-1-b(1). FIXED-FIXED, $\ell_1 = \ell/2$ - FIRST MODE - MODE SHAPE FACTOR



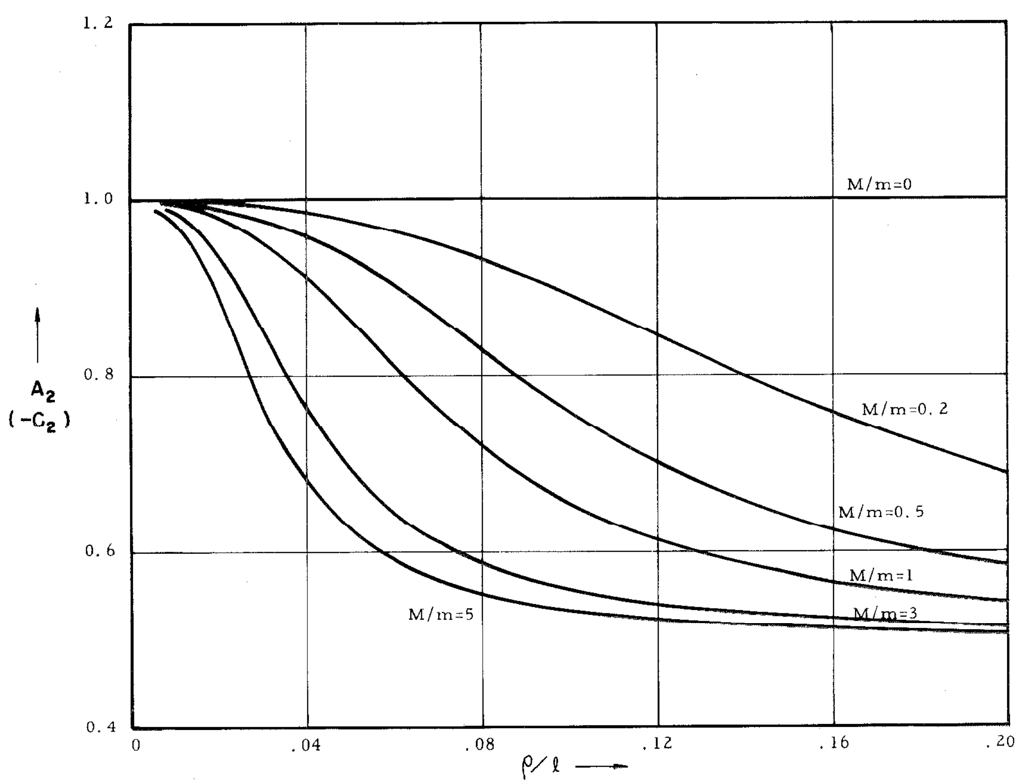
E-1-b(2). FIXED-FIXED, $\ell_1 = \ell/2$ - FIRST MODE - MODE SHAPE FACTOR



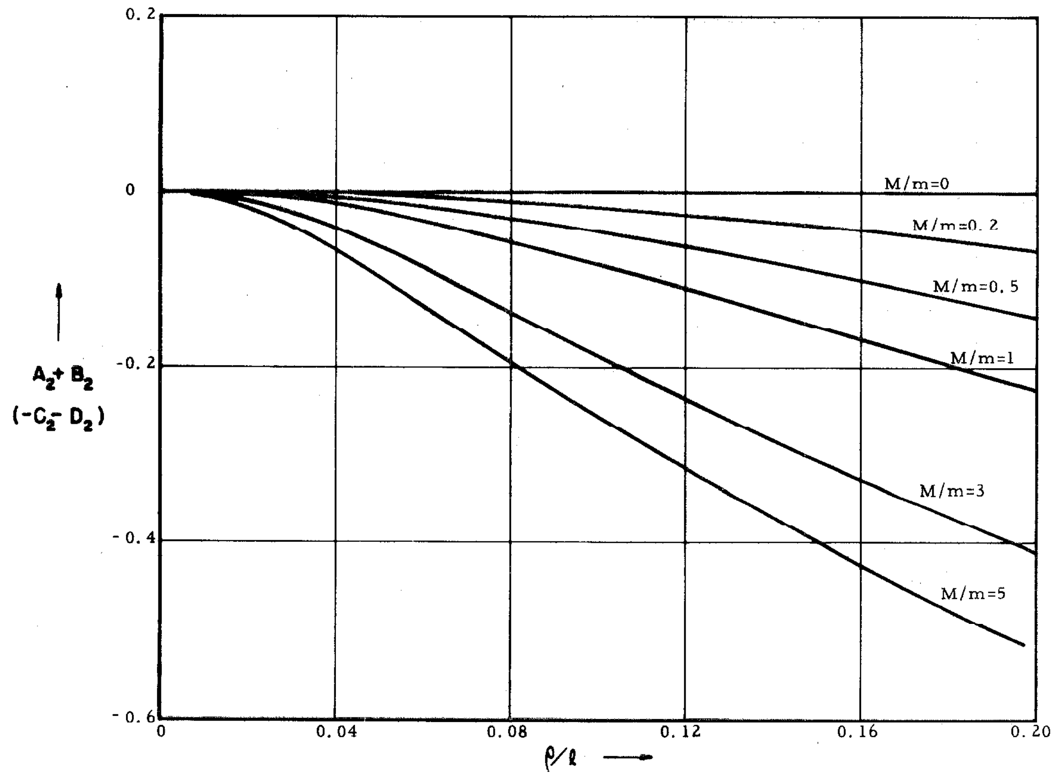
E-1-c. FIXED-FIXED, $\ell_1 = \ell/2$ - FIRST MODE - MODE PARTICIPATION FACTOR



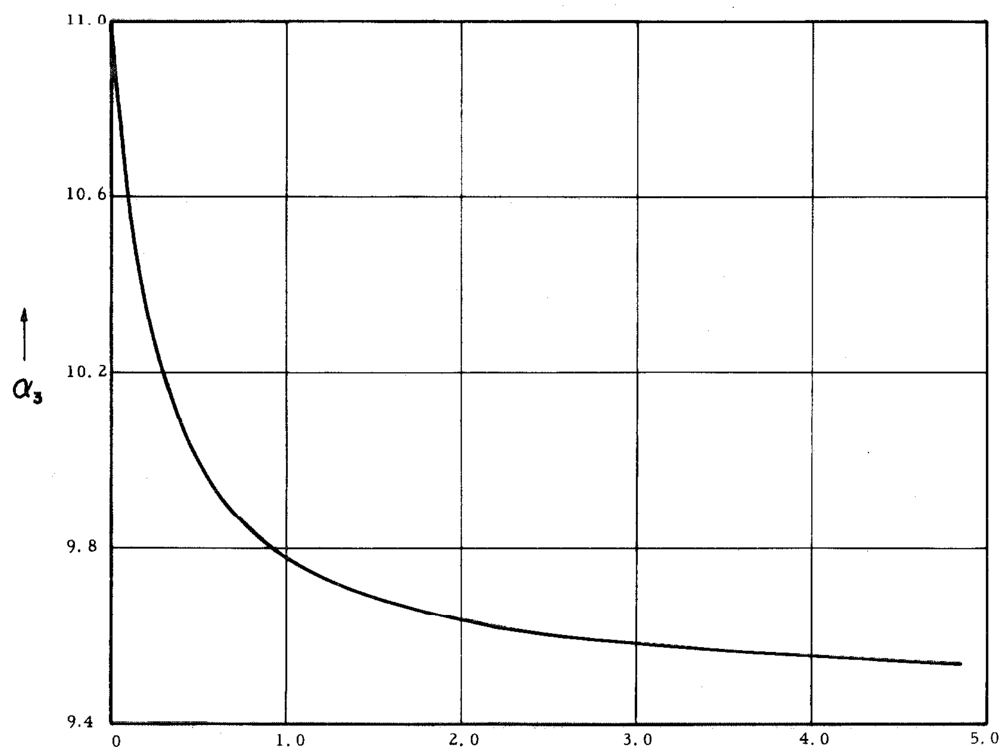
E-2-a. FIXED-FIXED, $\ell_1 = \ell/2$ - SEDOND MODE - FREQUENCY ROOT



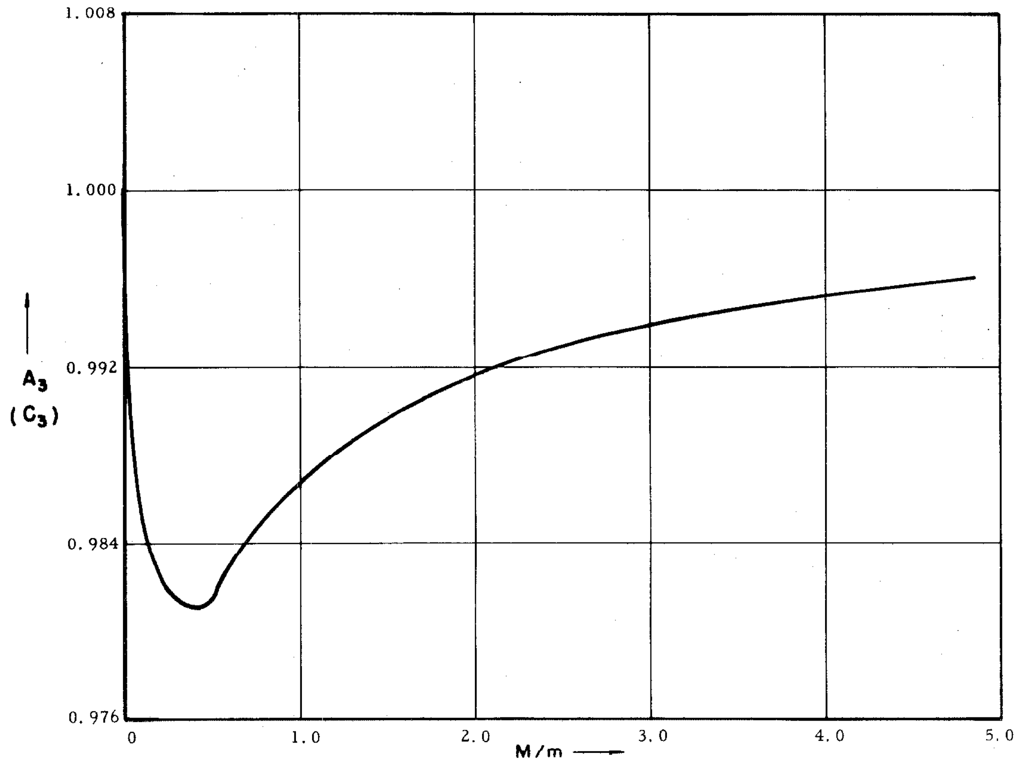
E-2-b(1). FIXED-FIXED, $\ell_1 = \ell/2$ - SECOND MODE - MODE SHAPE FACTOR



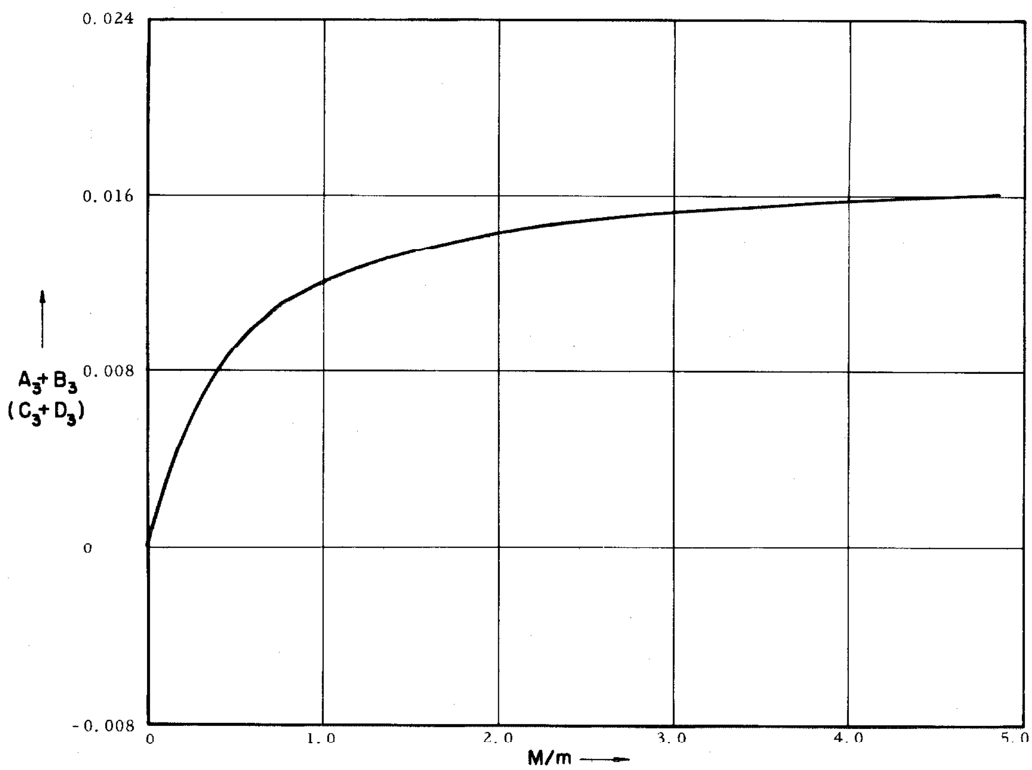
E-2-b(2). FIXED-FIXED, $\ell_1 = \ell/2$ - SECOND MODE - MODE SHAPE FACTOR



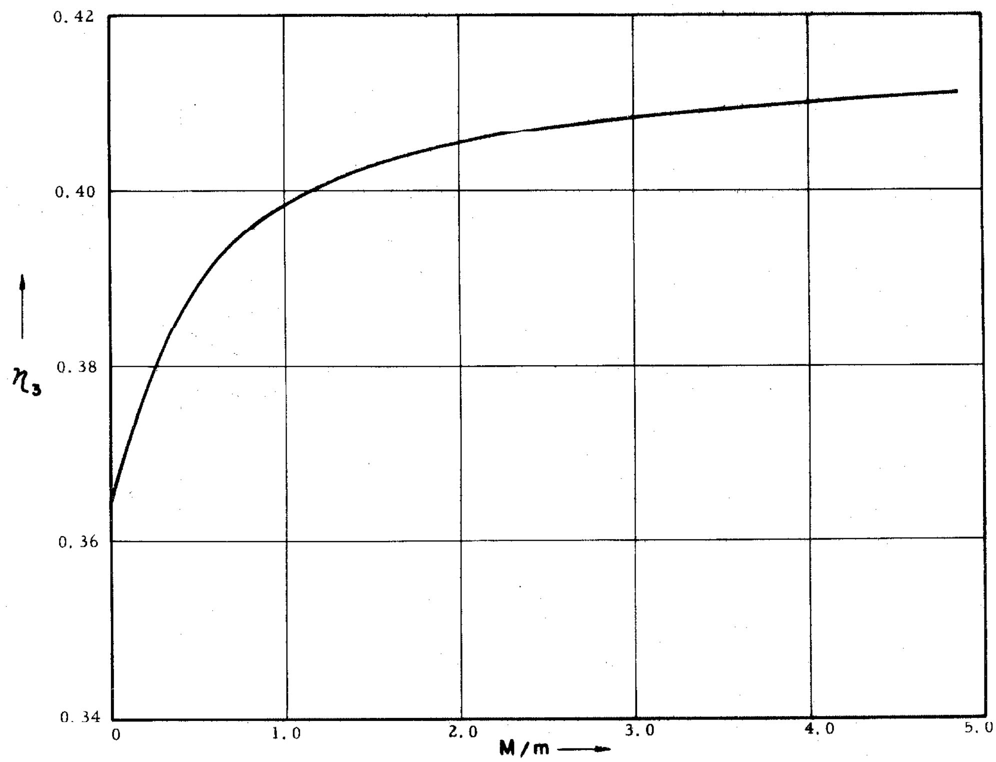
E-3-a. FIXED-FIXED, $\ell_1 = \ell/2$ - THIRD MODE - FREQUENCY ROOT



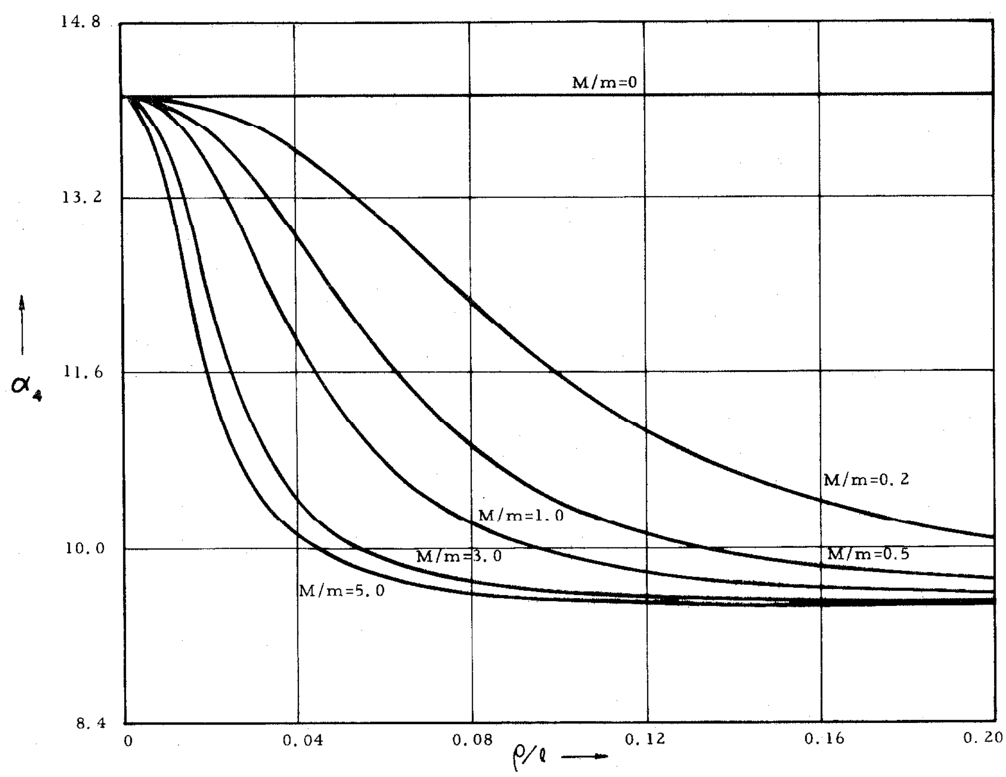
E-3-b(1). FIXED-FIXED, $\ell_1 = \ell/2$ - THIRD MODE - MODE SHAPE FACTOR



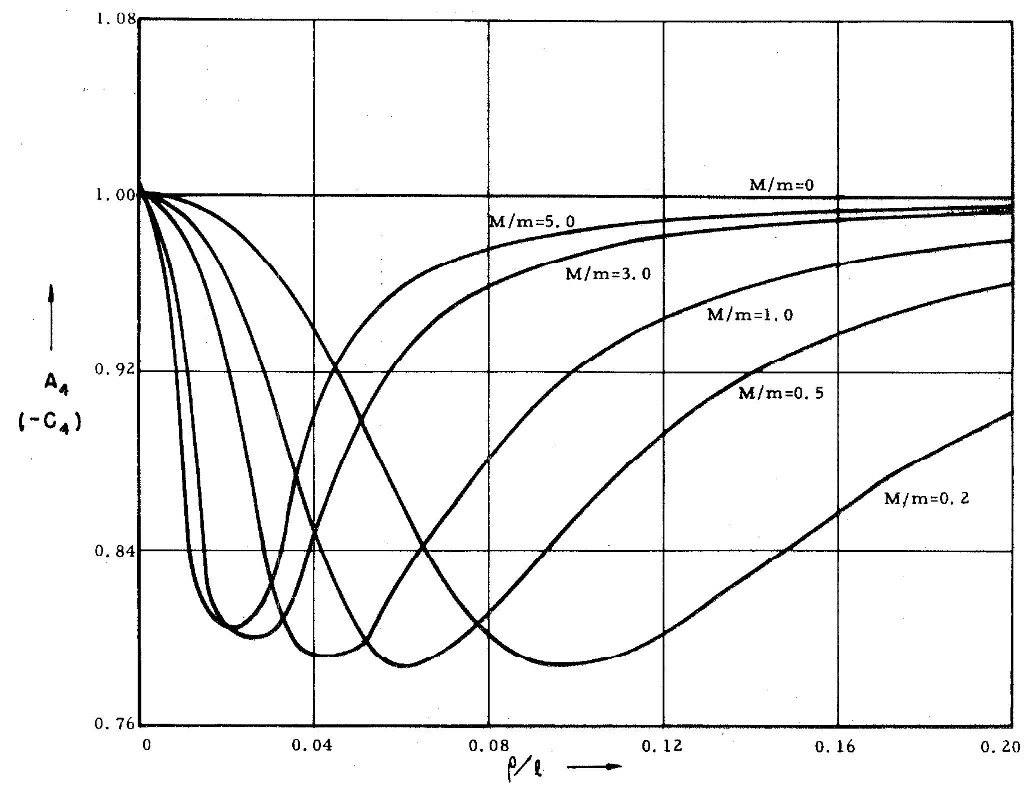
E-3-b(2). FIXED-FIXED, $\ell_1 = \ell/2$ - THIRD MODE - MODE SHAPE FACTOR



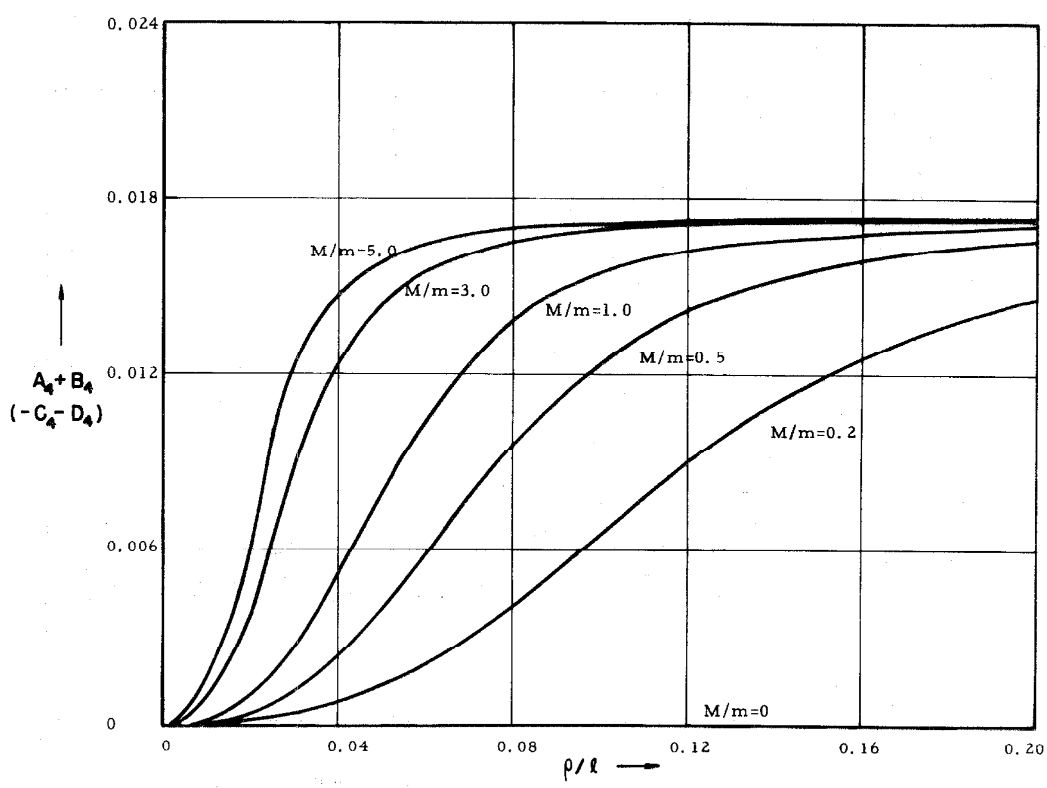
E-3-c. FIXED-FIXED, $\ell_1 = \ell/2$ - THIRD MODE - MODE PARTICIPATION FACTOR



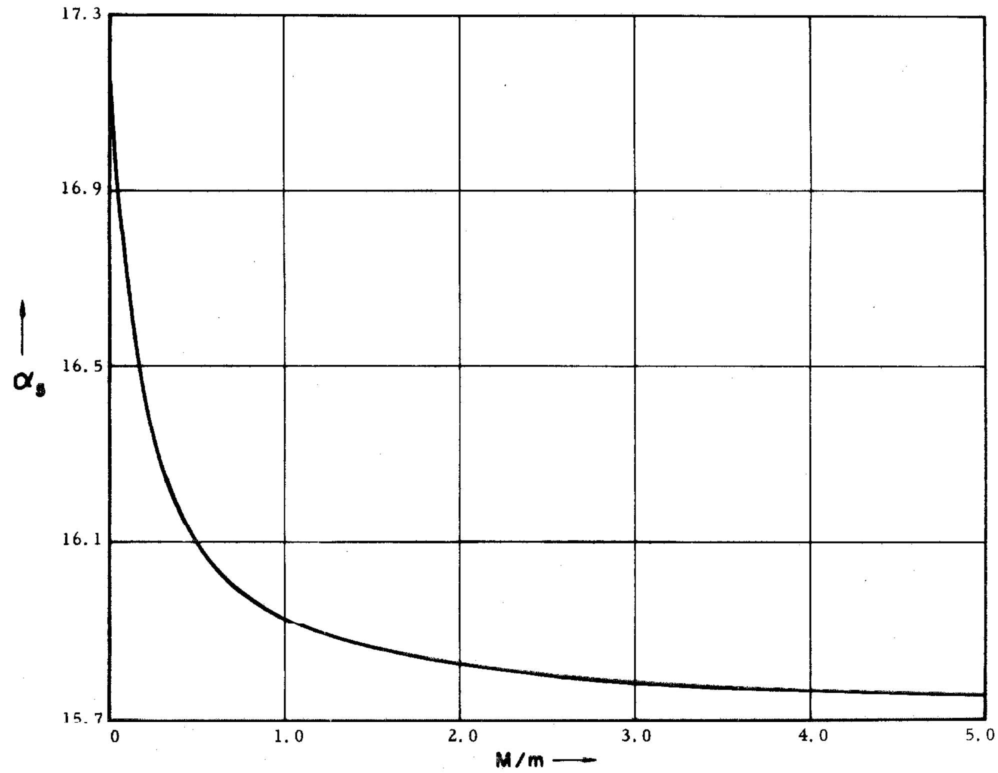
E-4-a. FIXED-FIXED, $\ell_1 = \ell/2$ - FOURTH MODE - FREQUENCY ROOT



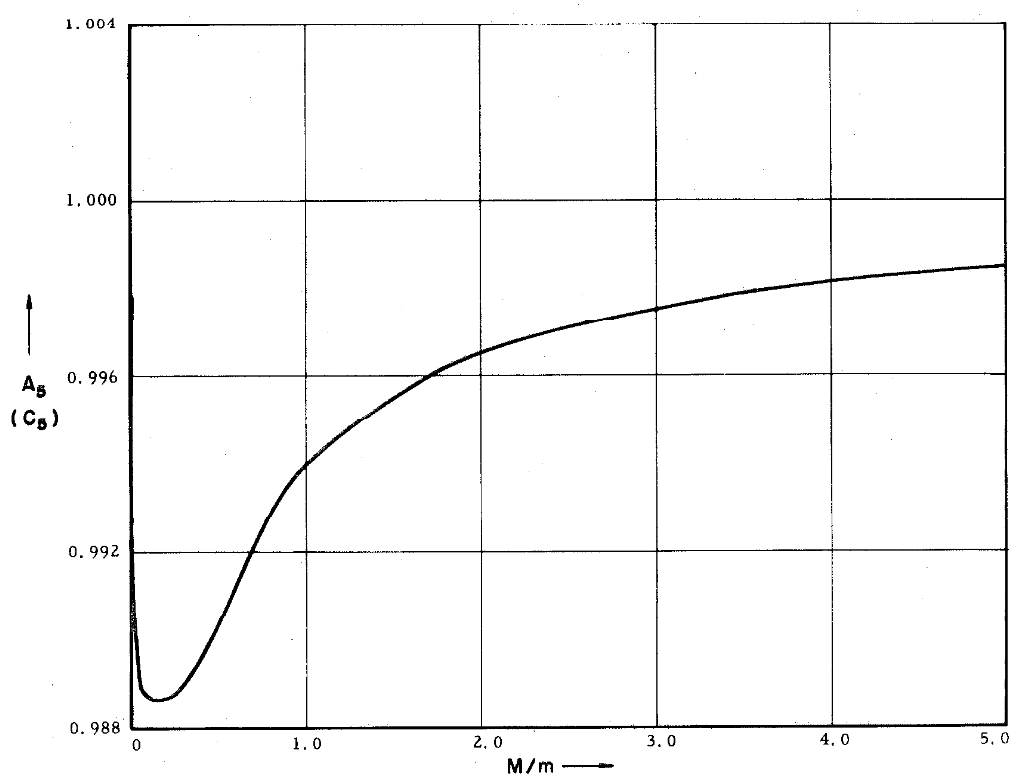
E-4-b(1). FIXED-FIXED, $\ell_1 = \ell/2$ - FOURTH MODE - MODE SHAPE FACTOR



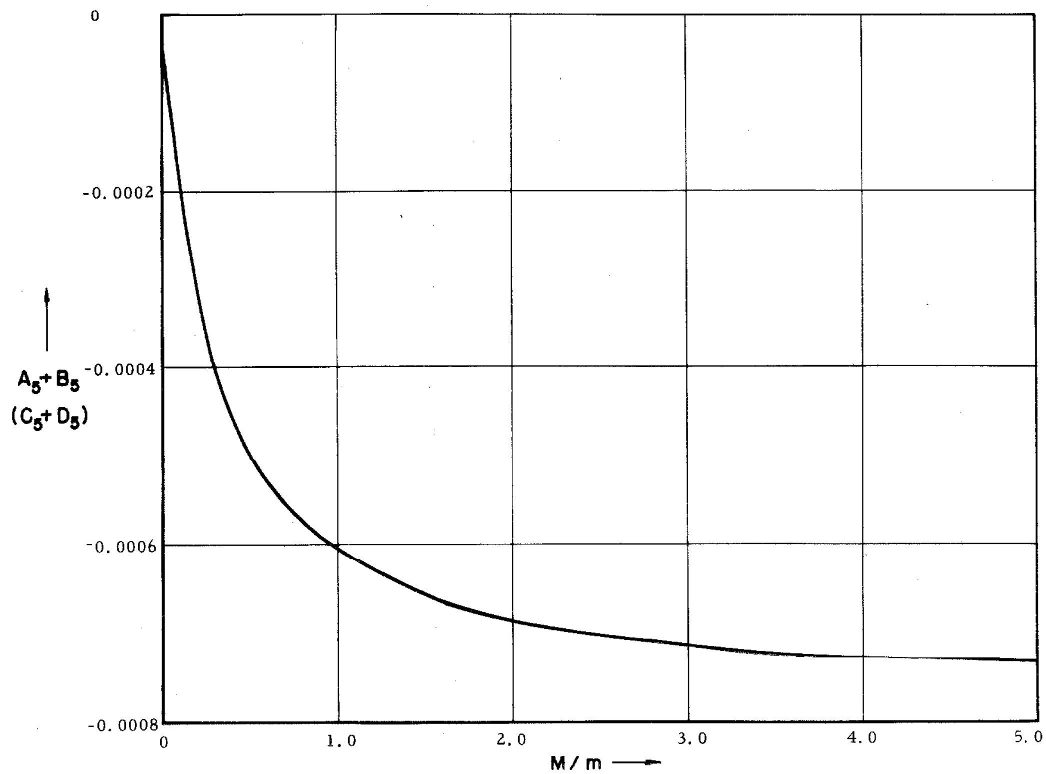
E-4-b(2). FIXED-FIXED, $\ell_1 = \ell/2$ - FOURTH MODE - MODE SHAPE FACTOR



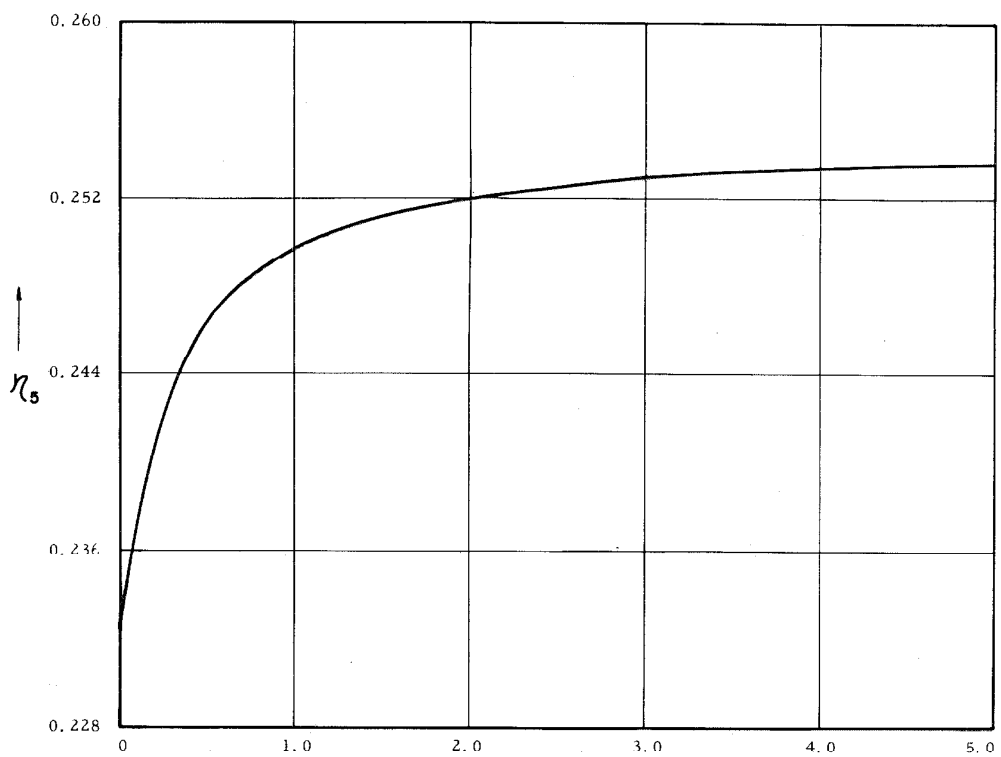
E-5-a. FIXED-FIXED, $\ell_1 = \ell/2$ - FIFTH MODE - FREQUENCY ROOT



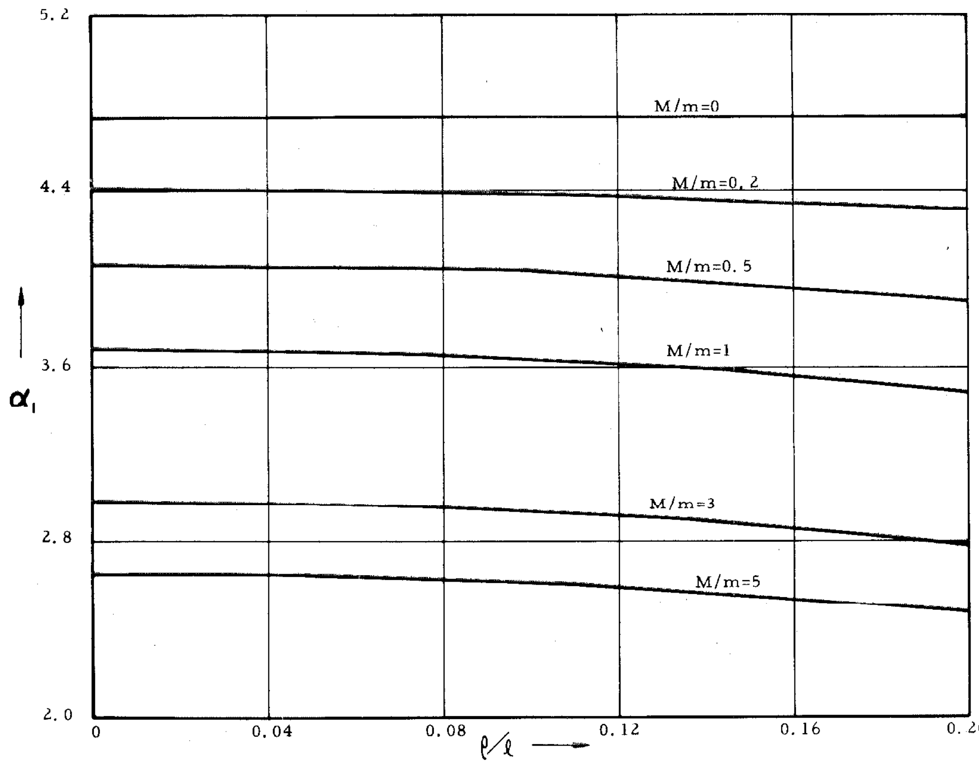
E-5-b(1). FIXED-FIXED, $\ell_1 = \ell/2$ - FIFTH MODE - MODE SHAPE FACTOR



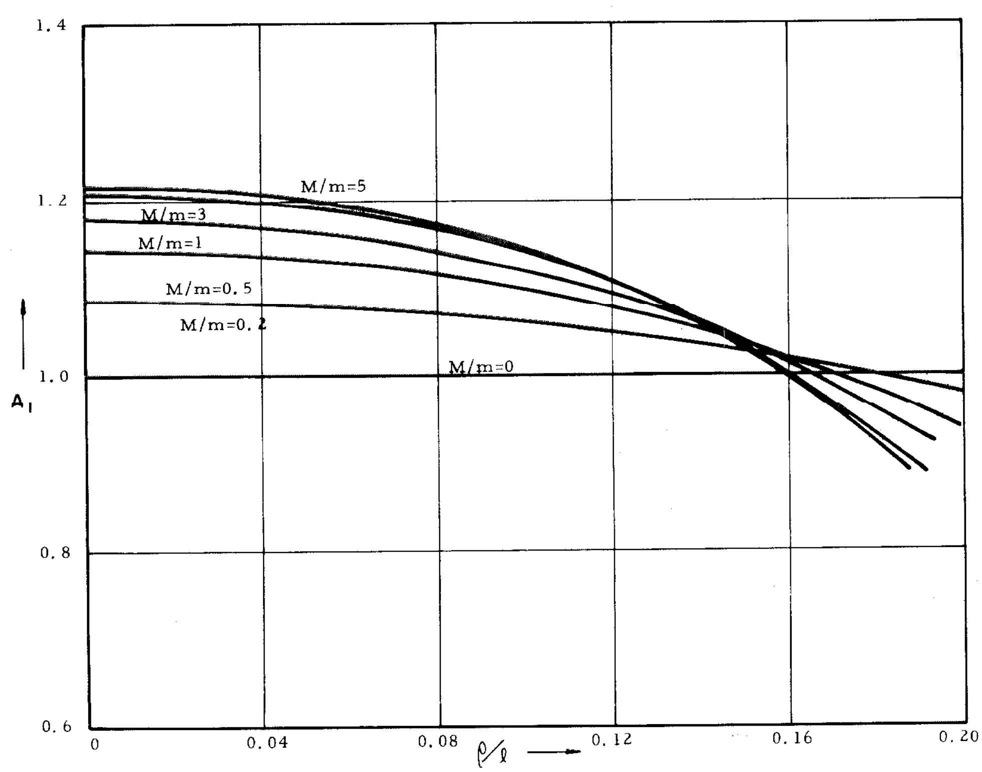
E-5-b(2). FIXED-FIXED, $\ell_1 = \ell/2$ - FIFTH MODE - MODE SHAPE FACTOR



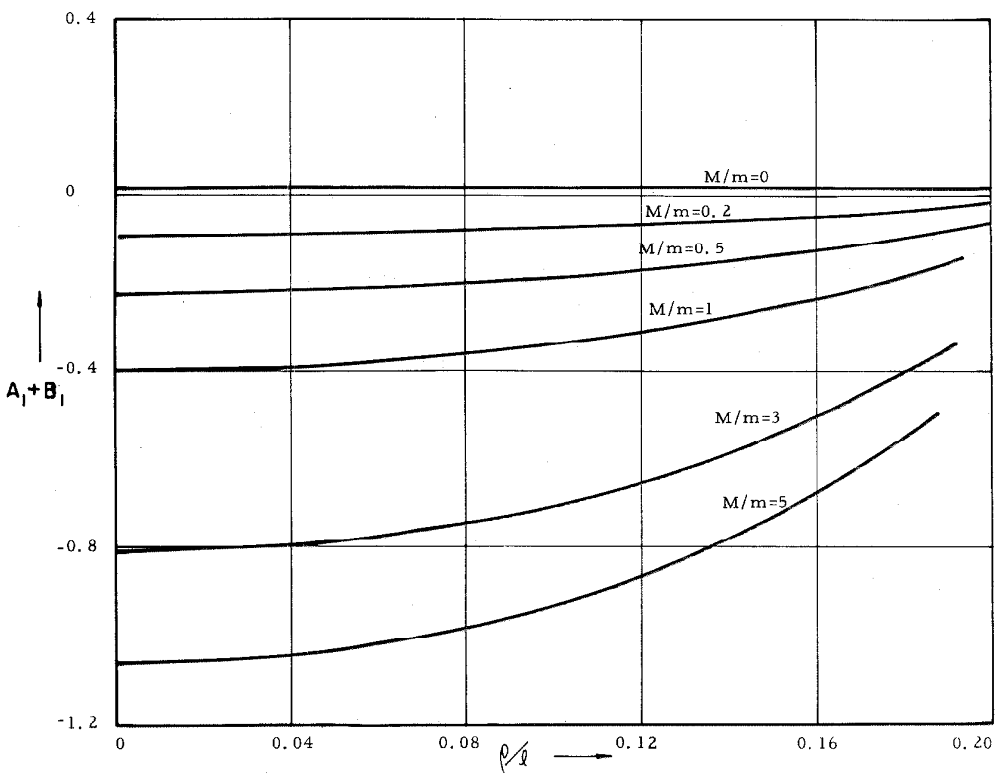
E-5-c. FIXED-FIXED, $\ell_1 = \ell/2$ - FIFTH MODE - MODE PARTICIPATION FACTOR



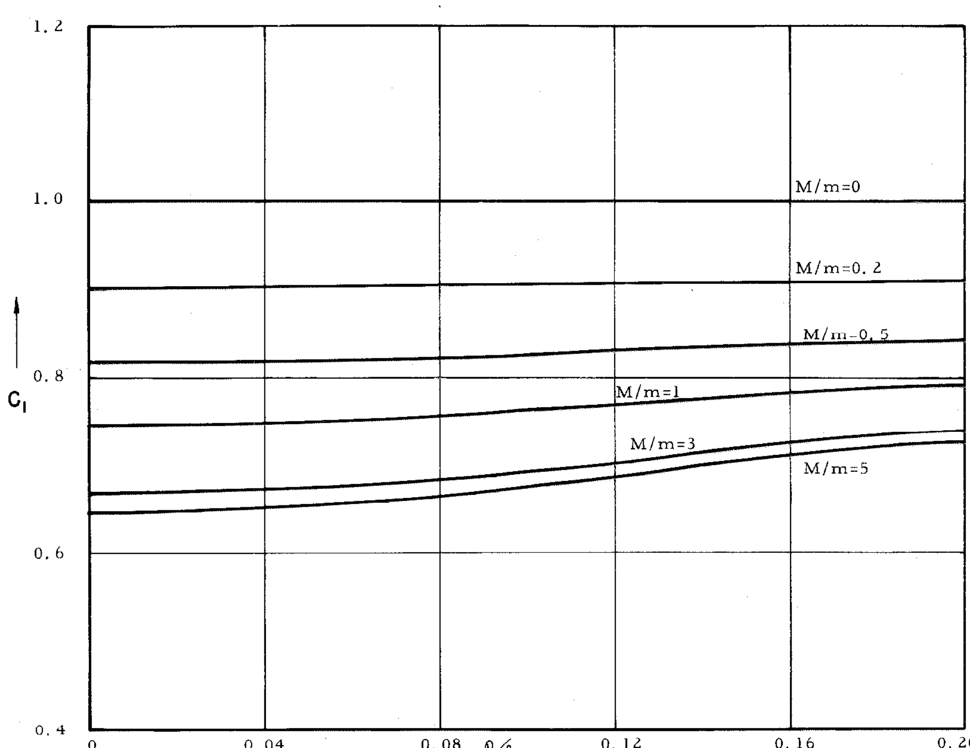
F-1-a. FIXED-FIXED, $\ell_1 = \ell/3$ - FIRST MODE - FREQUENCY ROOT



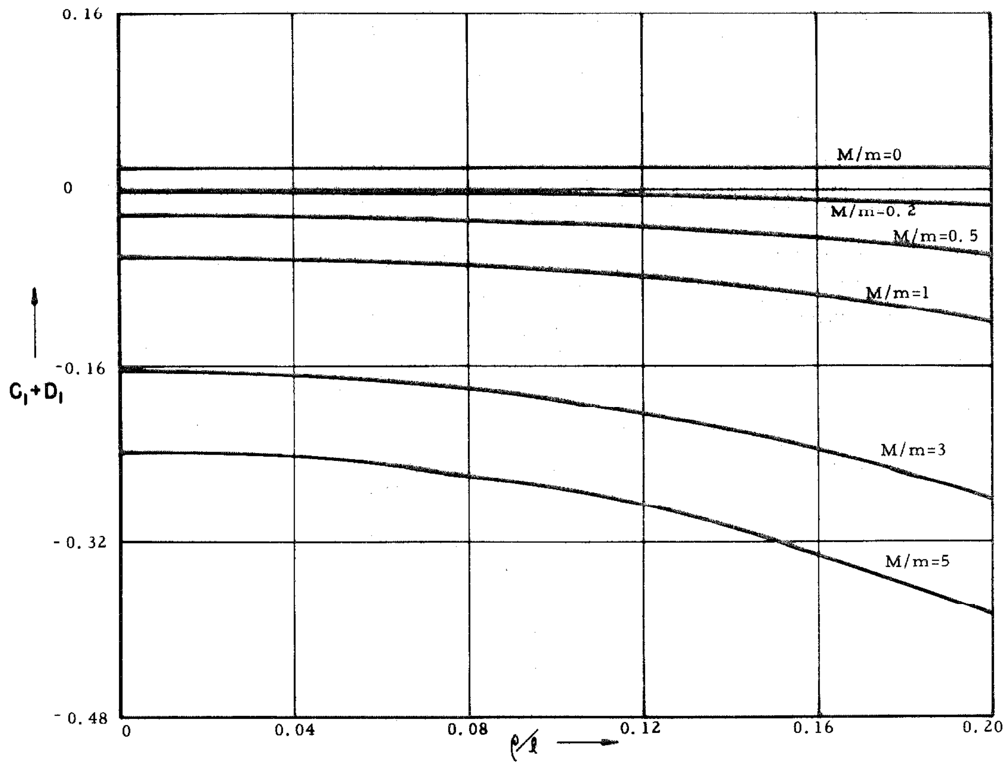
F-1-b(1). FIXED-FIXED, $\ell_1 = \ell/3$ - FIRST MODE - MODE SHAPE FACTOR



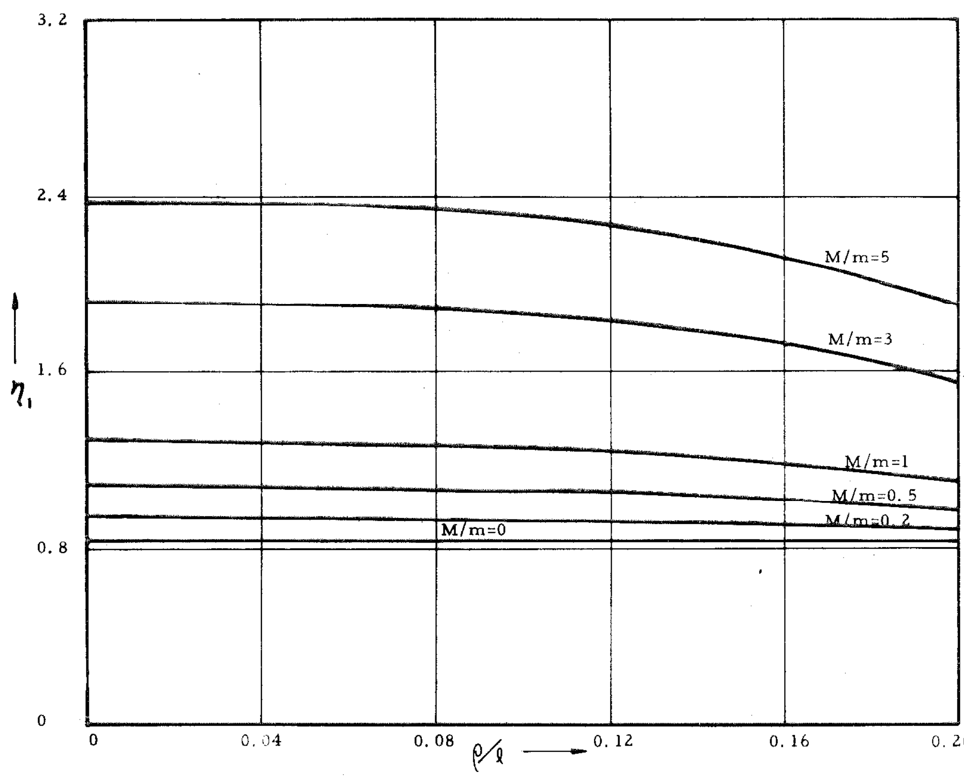
F-1-b(2). FIXED-FIXED, $\ell_1 = \ell/3$ - FIRST MODE - MODE SHAPE FACTOR



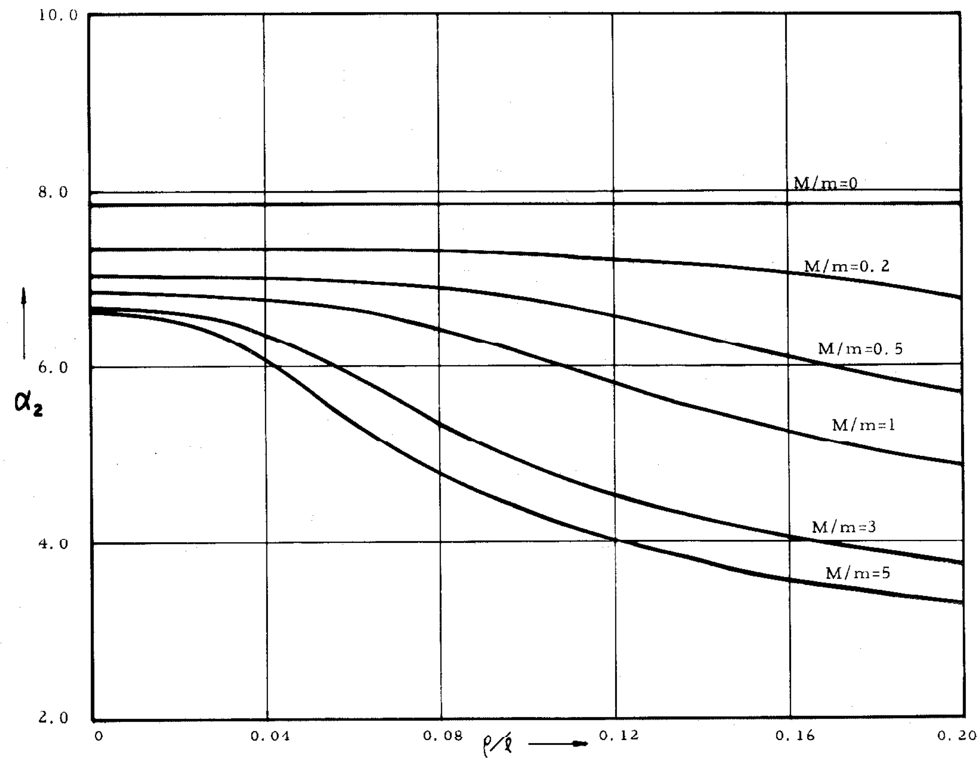
F-1-b(3). FIXED-FIXED, $\ell_1 = \ell/3$ - FIRST MODE - MODE SHAPE FACTOR



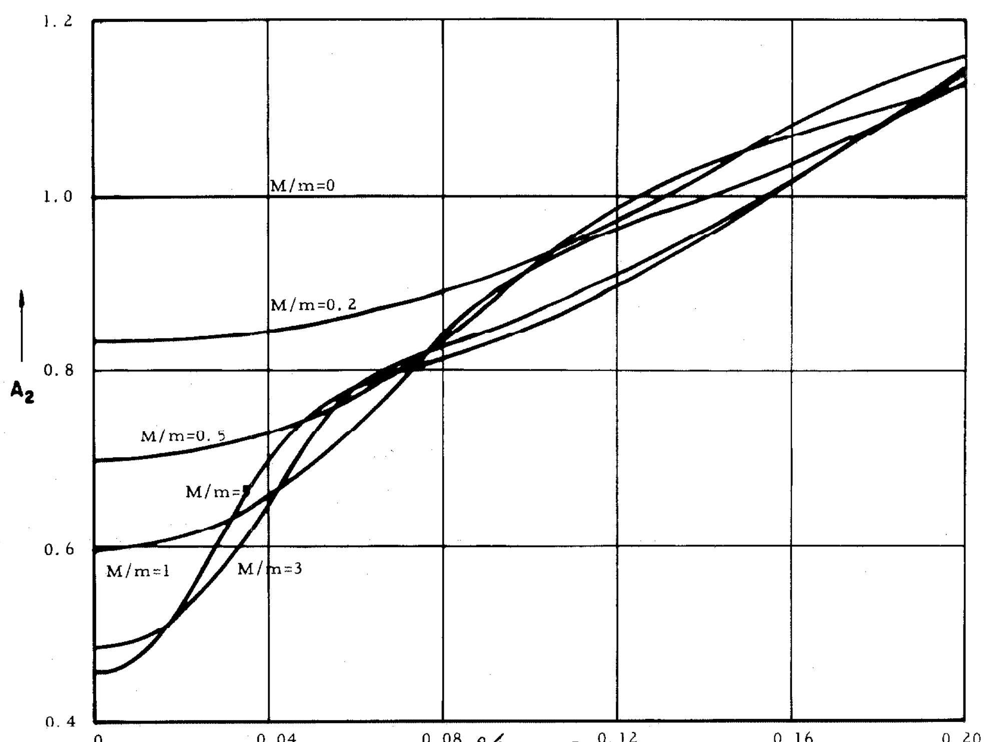
F-1-b(4). FIXED-FIXED, $\ell_1 = \ell/3$ - FIRST MODE - MODE SHAPE FACTOR



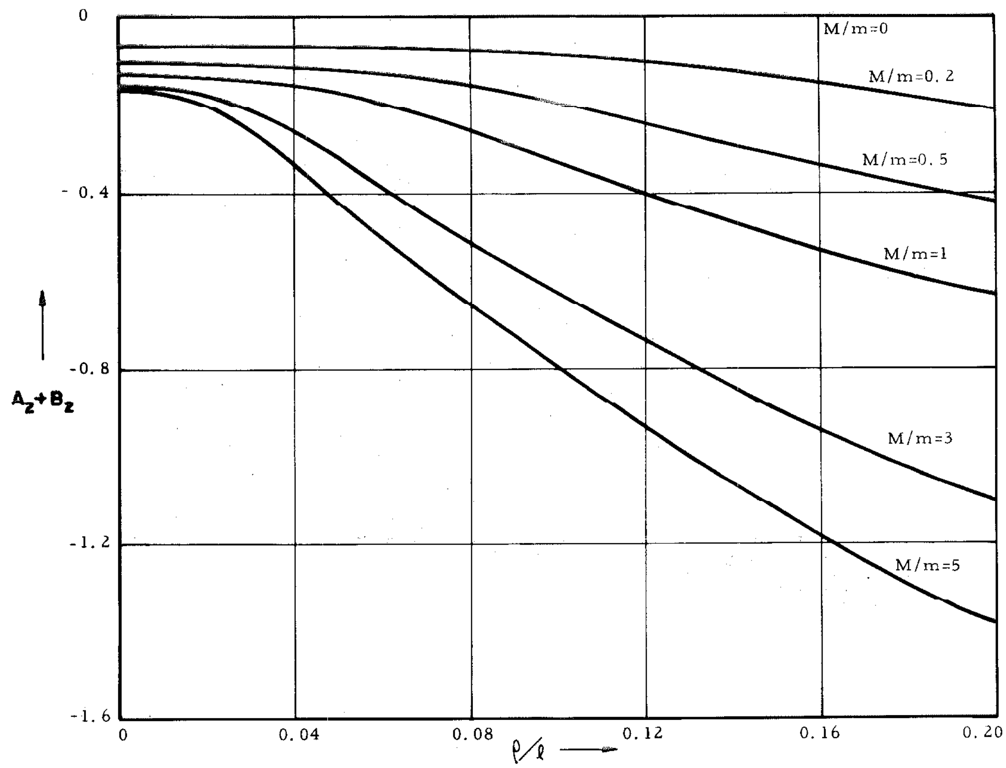
F-1-c. FIXED-FIXED, $\ell_1 = \ell/3$ - FIRST MODE - MODE PARTICIPATION FACTOR



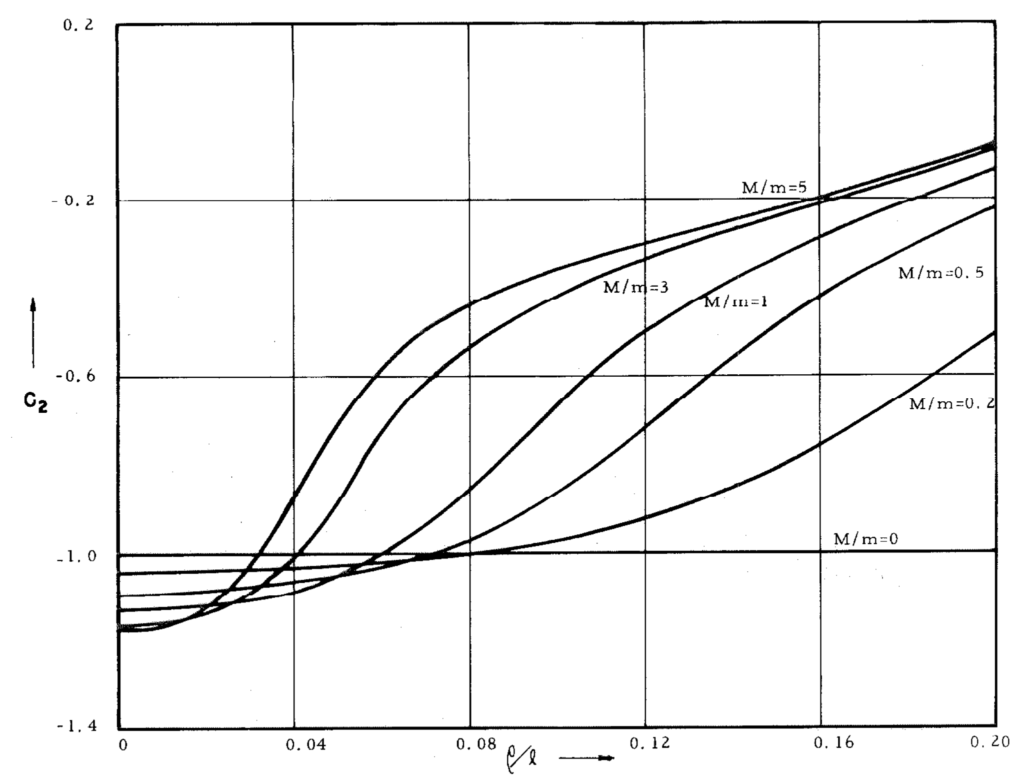
F-2-a. FIXED-FIXED, $\ell_1 = \ell/3$ - SECOND MODE - FREQUENCY ROOT



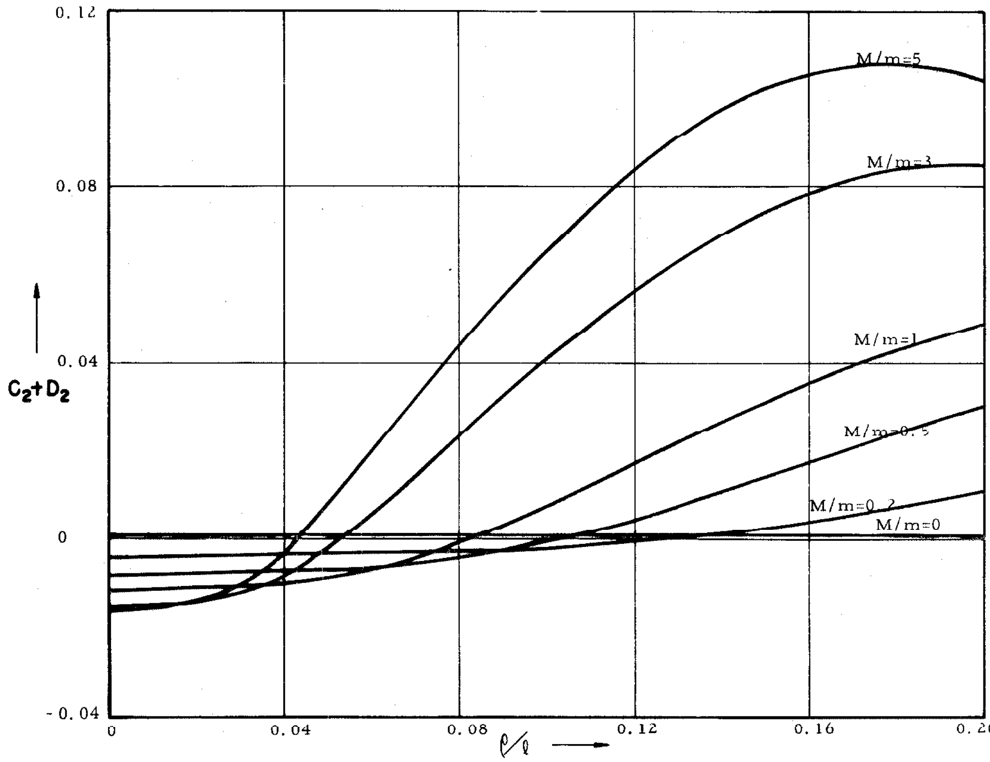
F-2-b(1). FIXED-FIXED, $\ell_1 = \ell/3$ - SECOND MODE - MODE SHAPE FACTOR



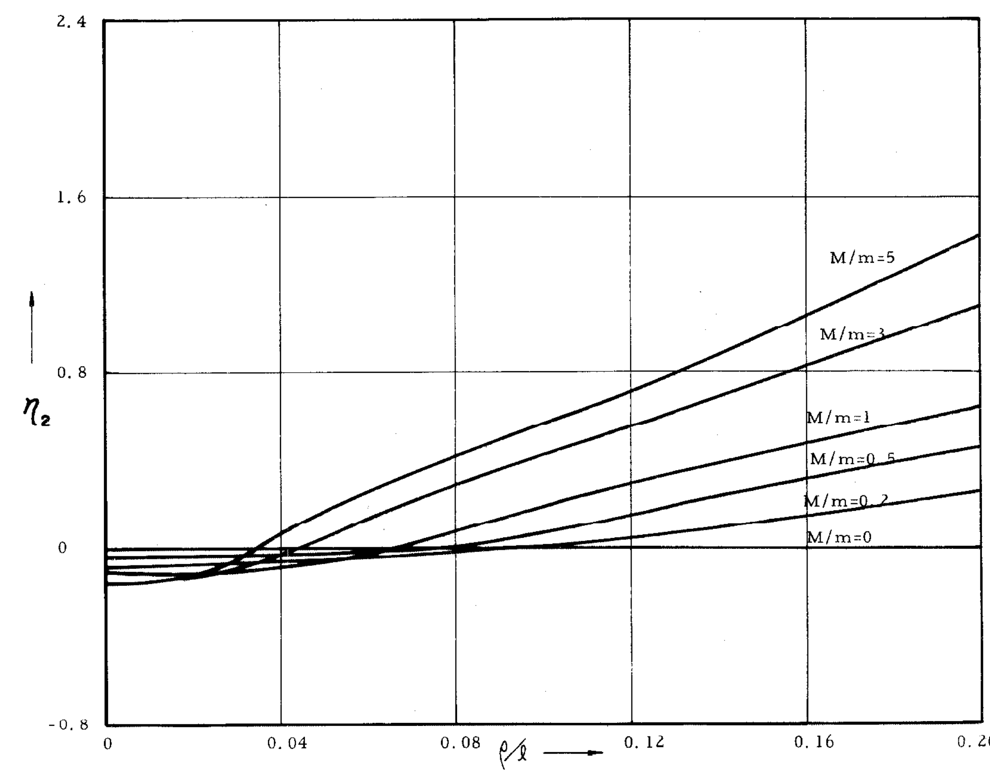
F-2-b(2). FIXED-FIXED, $\ell_1 = \ell/3$ - SECOND MODE - MODE SHAPE FACTOR



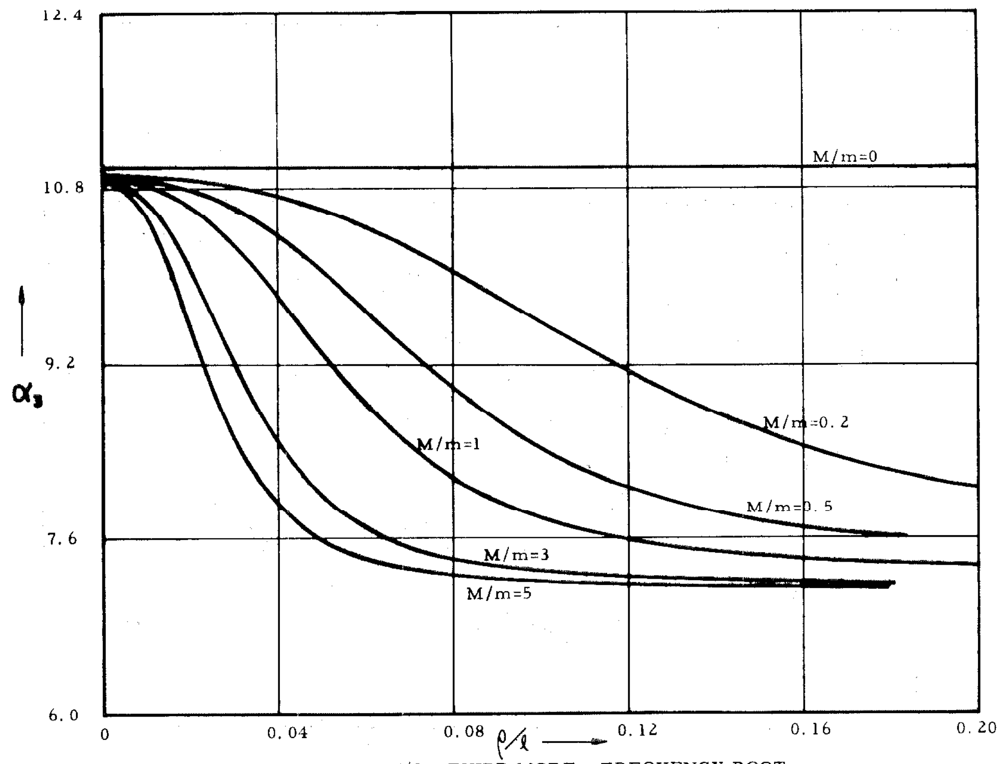
F-2-b(3). FIXED-FIXED, $\ell_1 = \ell/3$ - SECOND MODE - MODE SHAPE FACTOR



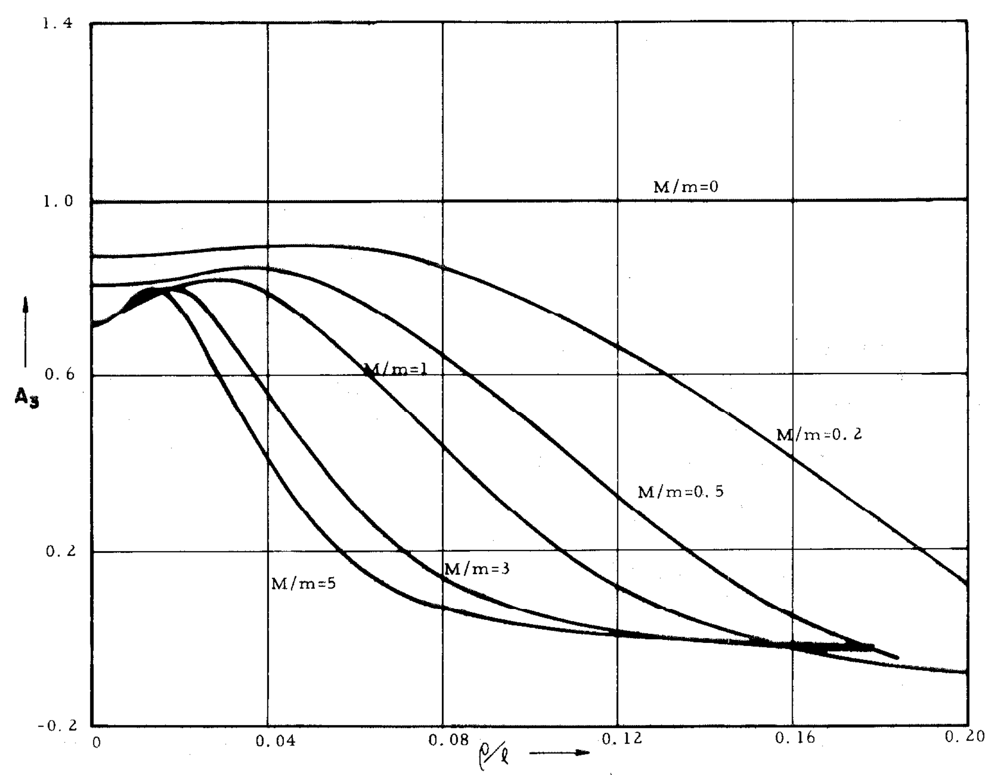
F-2-b(4). FIXED-FIXED, $\ell_1 = \ell/3$ - SECOND MODE - MODE SHAPE FACTOR



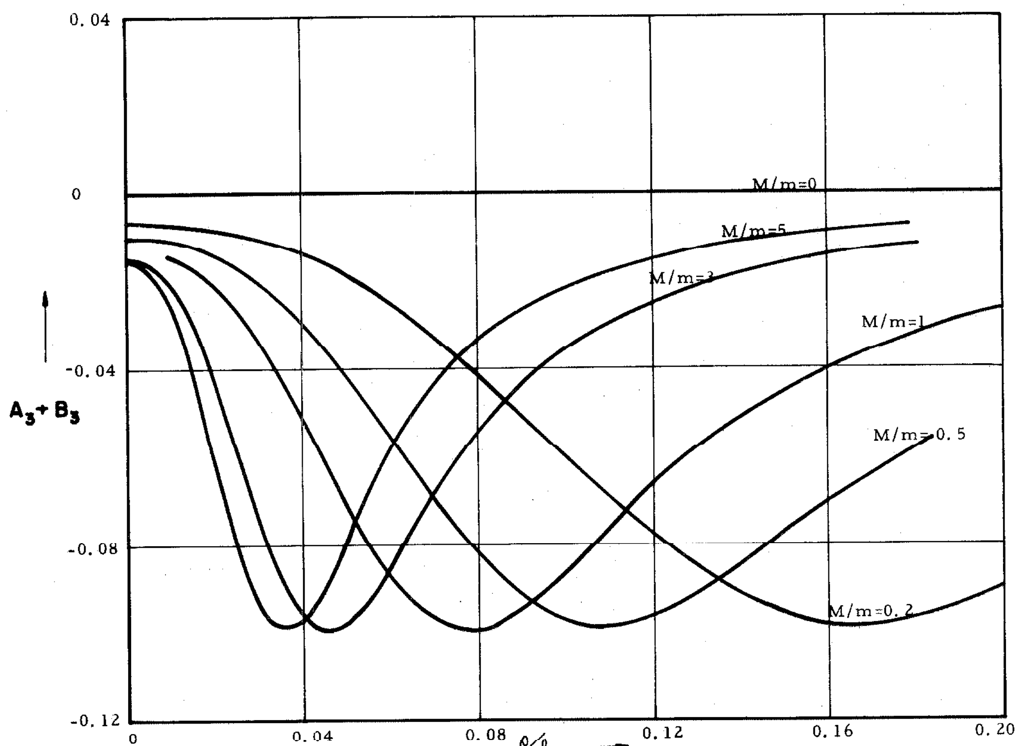
F-2-c. FIXED-FIXED, $\ell_1 = \ell/3$ - SECOND MODE - MODE PARTICIPATION FACTOR



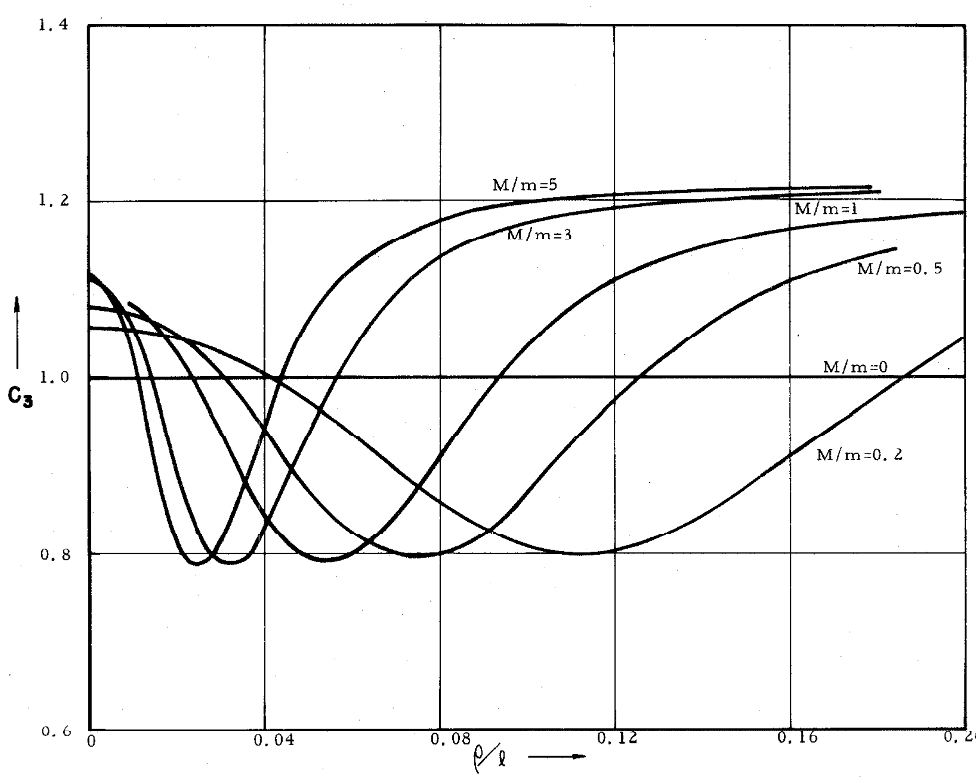
F-3-a. FIXED-FIXED, $\ell_1 = \ell/3$ - THIRD MODE - FREQUENCY ROOT



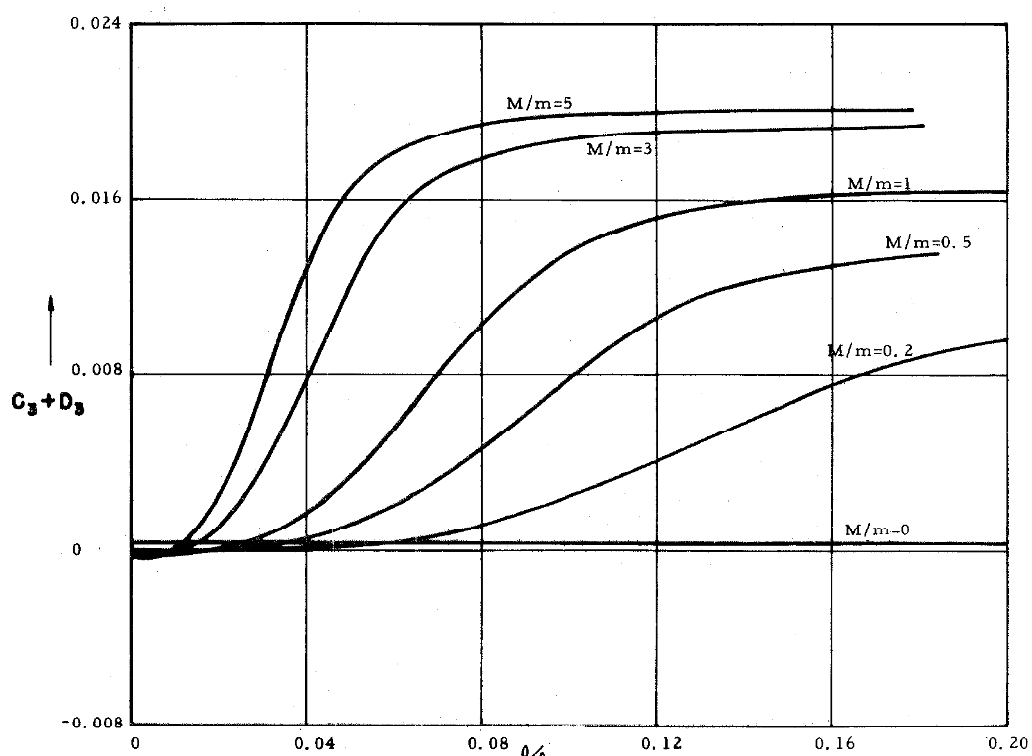
F-3-b(1). FIXED-FIXED, $\ell_1 = \ell/3$ - THIRD MODE - MODE SHAPE FACTOR



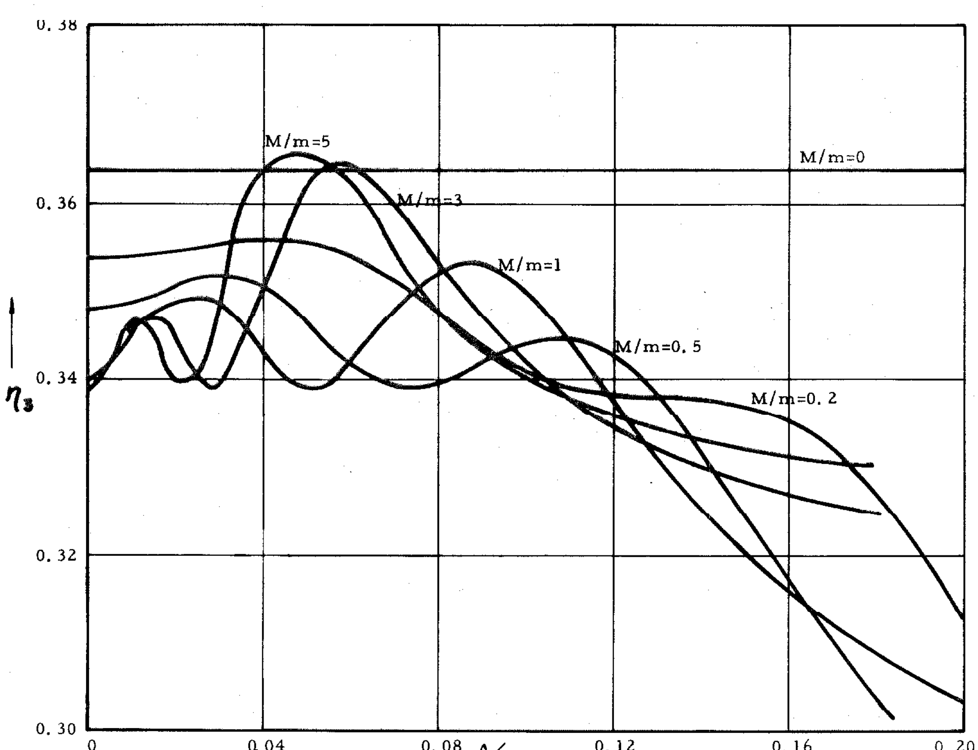
F-3-b(2). FIXED-FIXED, $\ell_1 = \ell/3$ - THIRD MODE - MODE SHAPE FACTOR



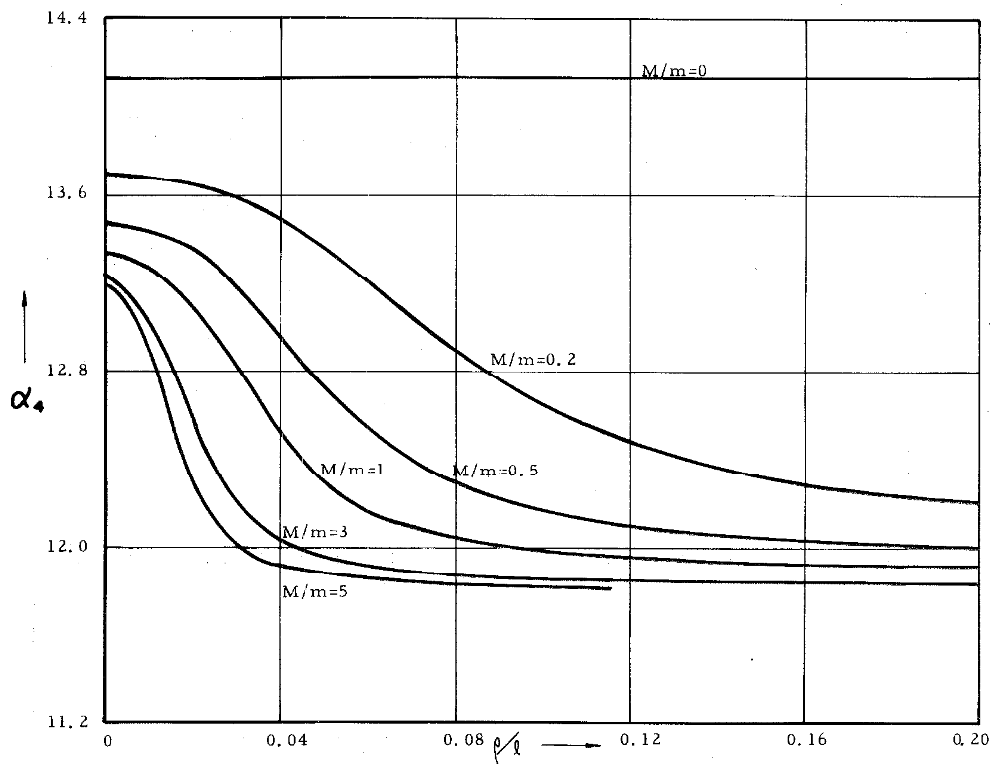
F-3-b(3). FIXED-FIXED, $\ell_1 = \ell/3$ - THIRD MODE - MODE SHAPE FACTOR



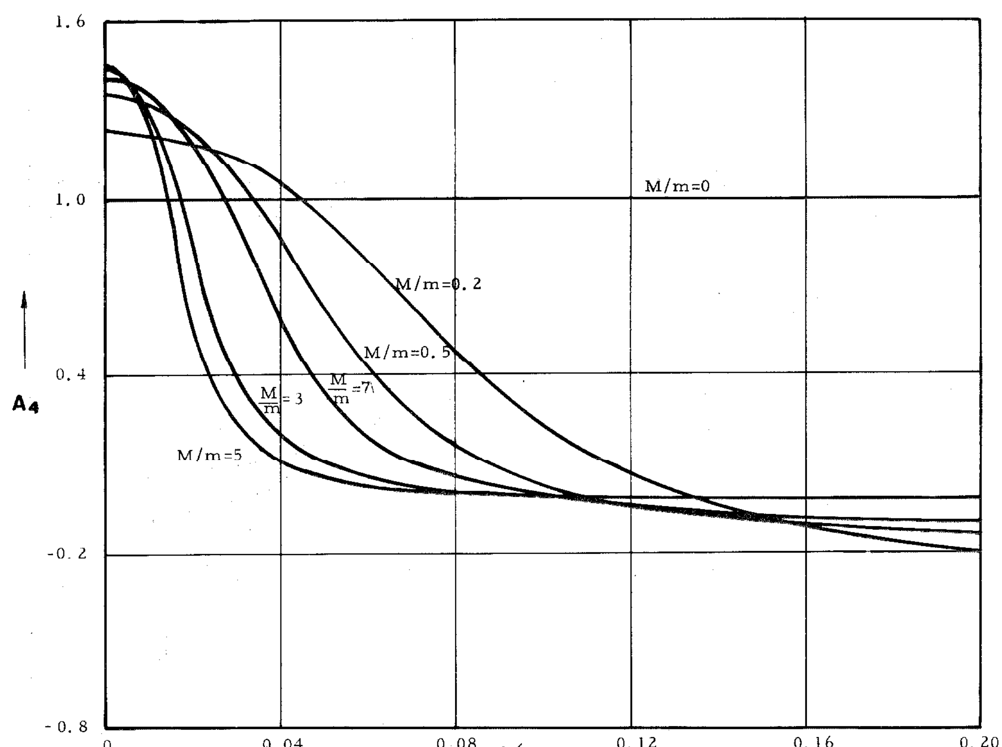
F-3-b(4). FIXED-FIXED, $\ell_1 = \ell/3$ - THIRD MODE - MODE SHAPE FACTOR



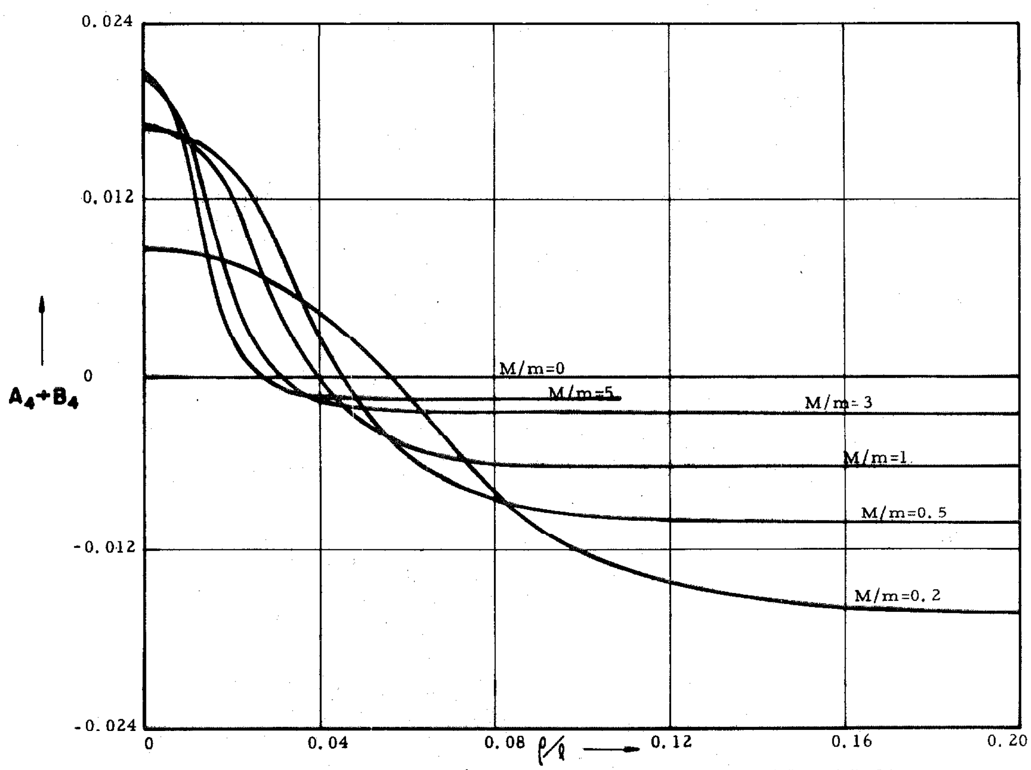
F-3-c . FIXED-FIXED, $\ell_1 = \ell/3$ - THIRD MODE - MODE PARTICIPATION FACTOR



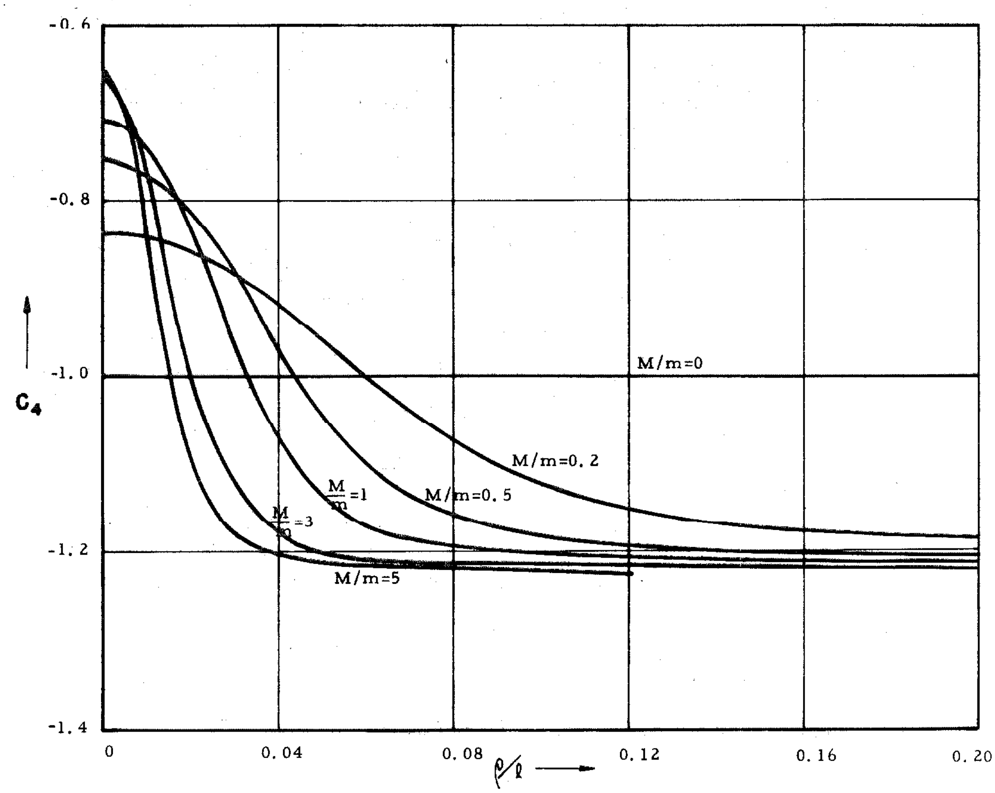
F-4-a. FIXED-FIXED, $\ell_1 = \ell/3$ - FOURTH MODE - FREQUENCY ROOT



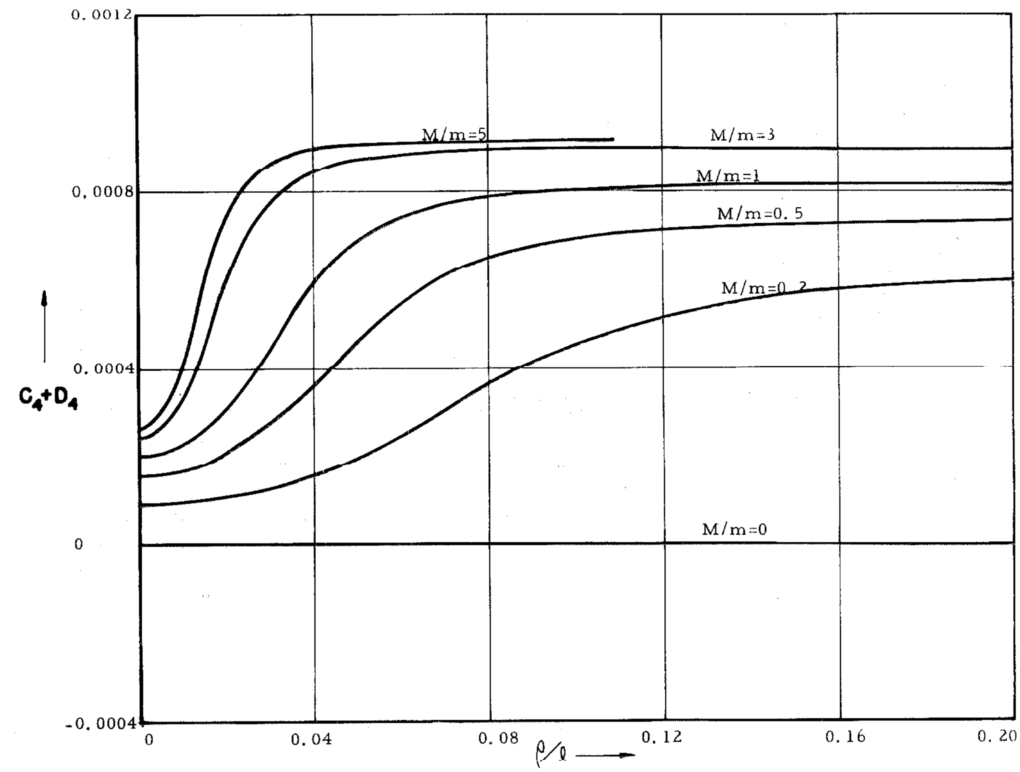
F-4-b(1). FIXED-FIXED, $\ell_1 = \ell/3$ - FOURTH MODE - MODE SHAPE FACTOR



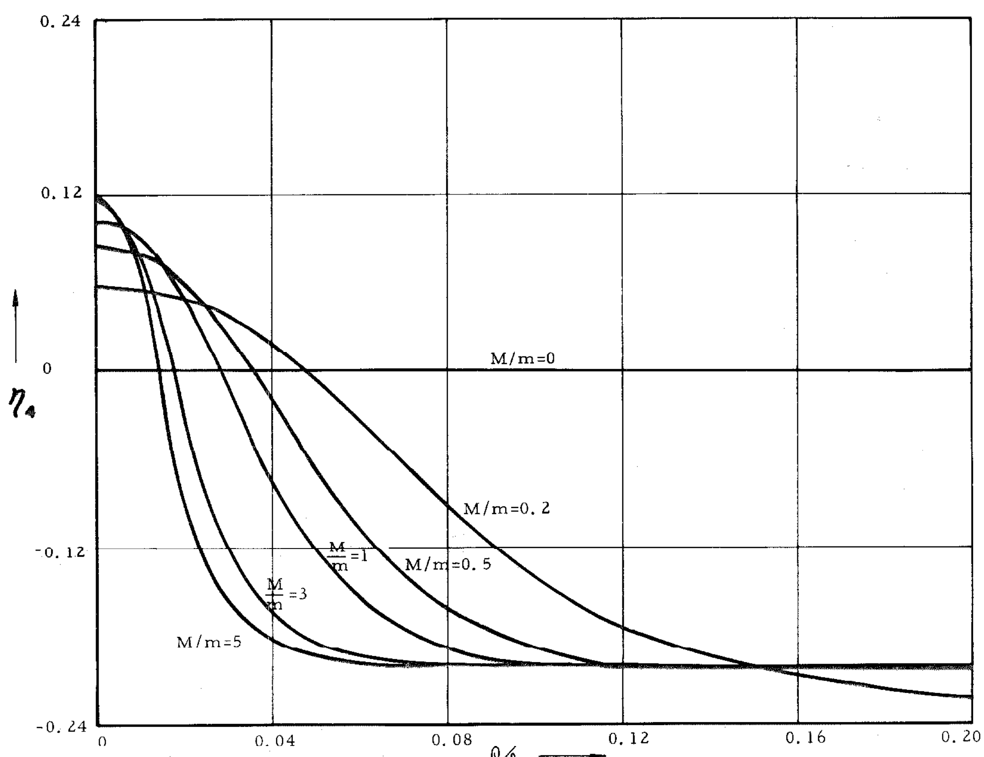
F-4-b(2). FIXED-FIXED, $\ell_1 = \ell/3$ - FOURTH MODE - MODE SHAPE FACTOR



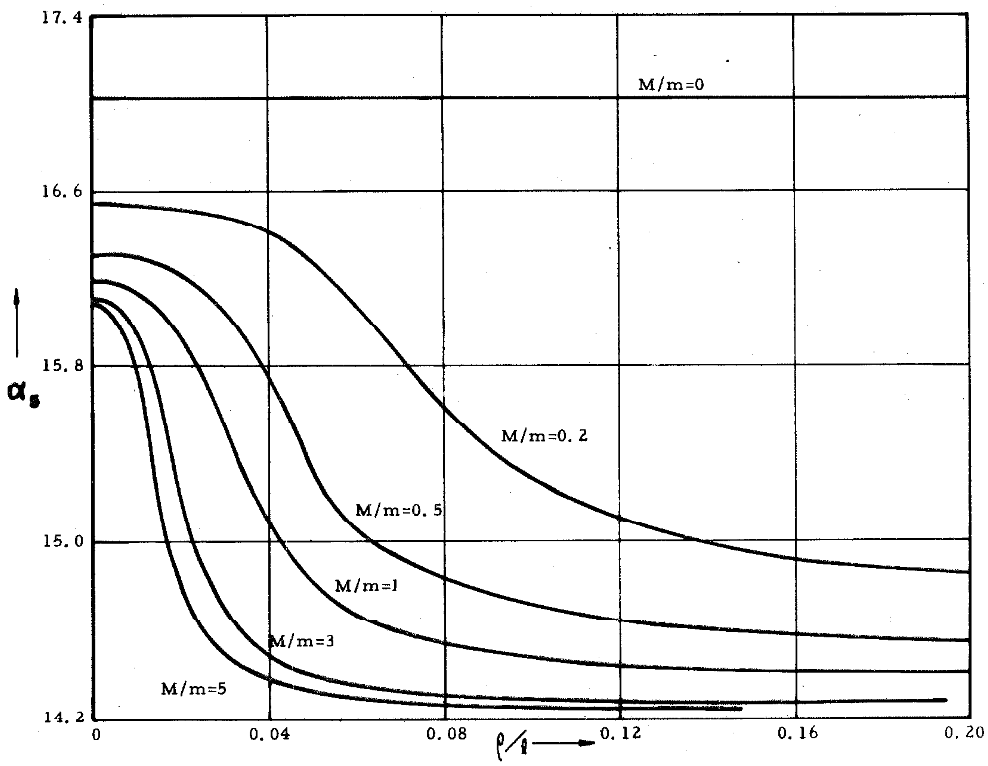
F-4-b(3). FIXED-FIXED, $\ell_1 = \ell/3$ - FOURTH MODE - MODE SHAPE FACTOR



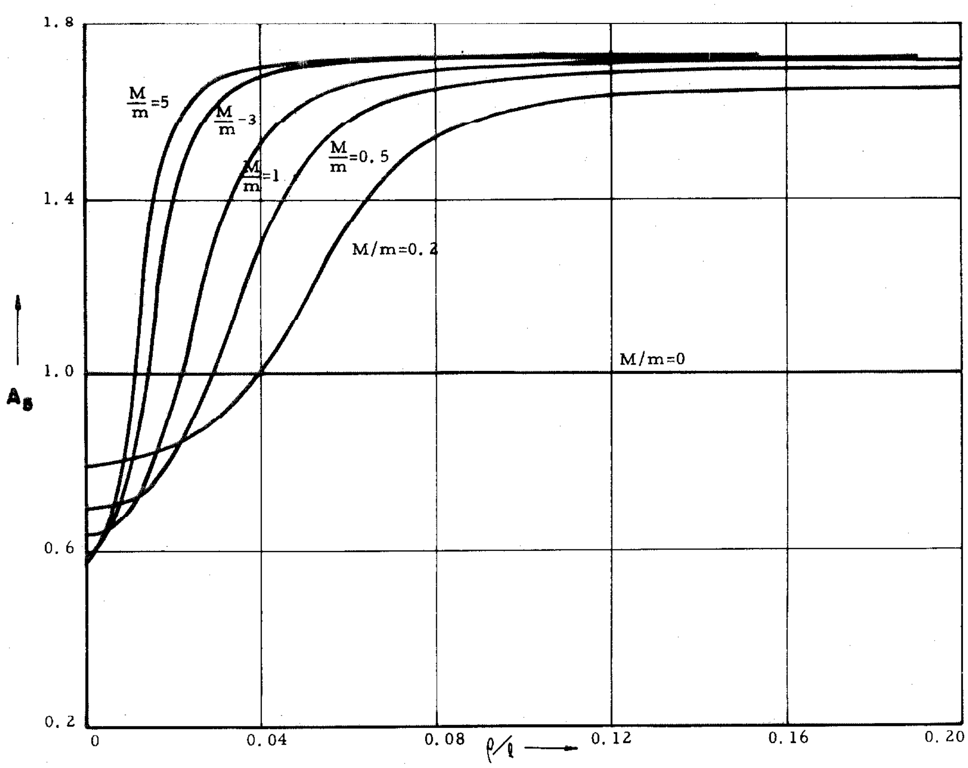
F-4-b(4). FIXED-FIXED, $\ell_1 = \ell/3$ - FOURTH MODE - MODE SHAPE FACTOR



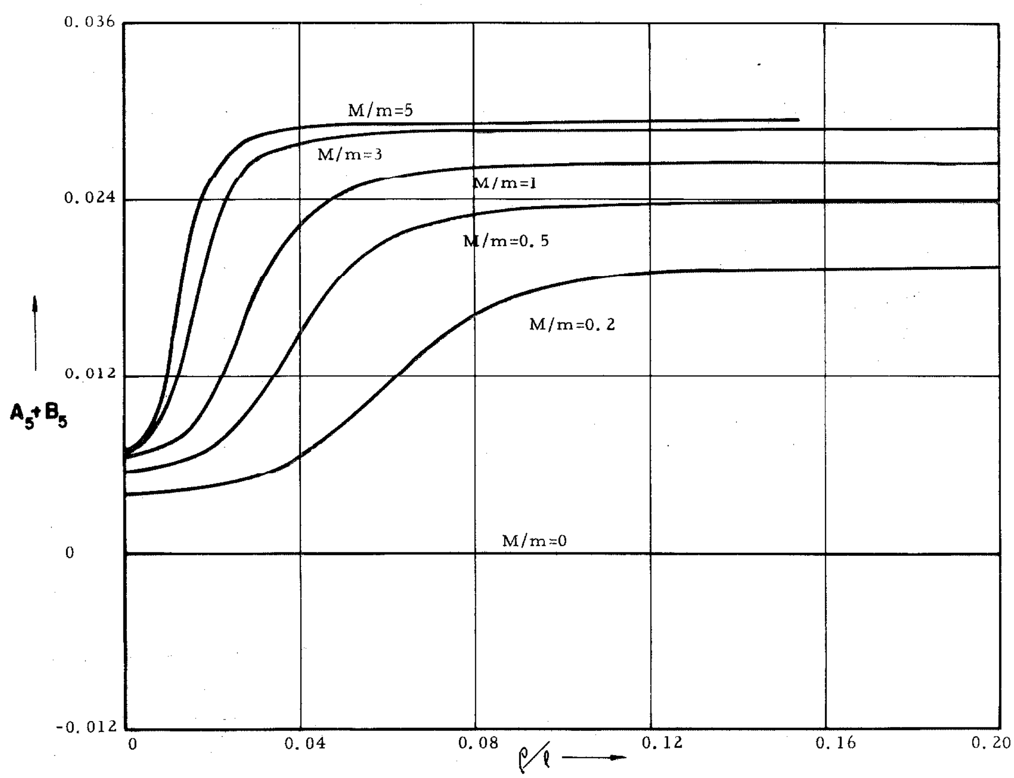
F-4-c. FIXED-FIXED, $\ell_1 = \ell/3$ - FOURTH MODE - MODE PARTICIPATION FACTOR



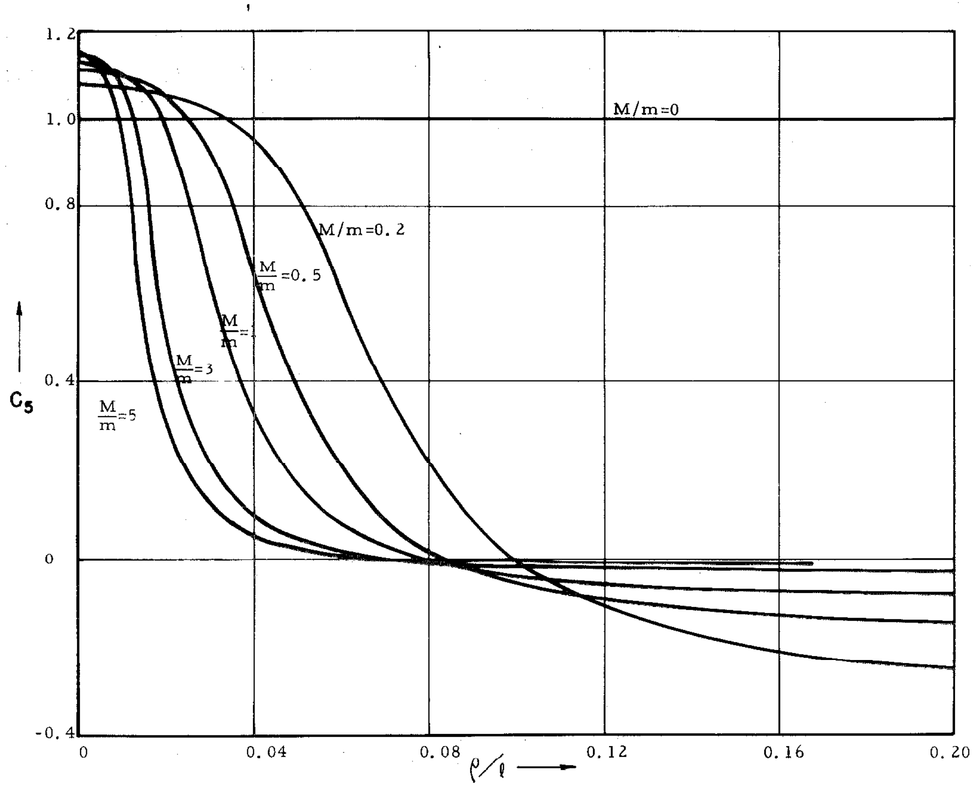
F-5-a. FIXED-FIXED, $\ell_1 = \ell/3$ - FIFTH MODE - FREQUENCY ROOT



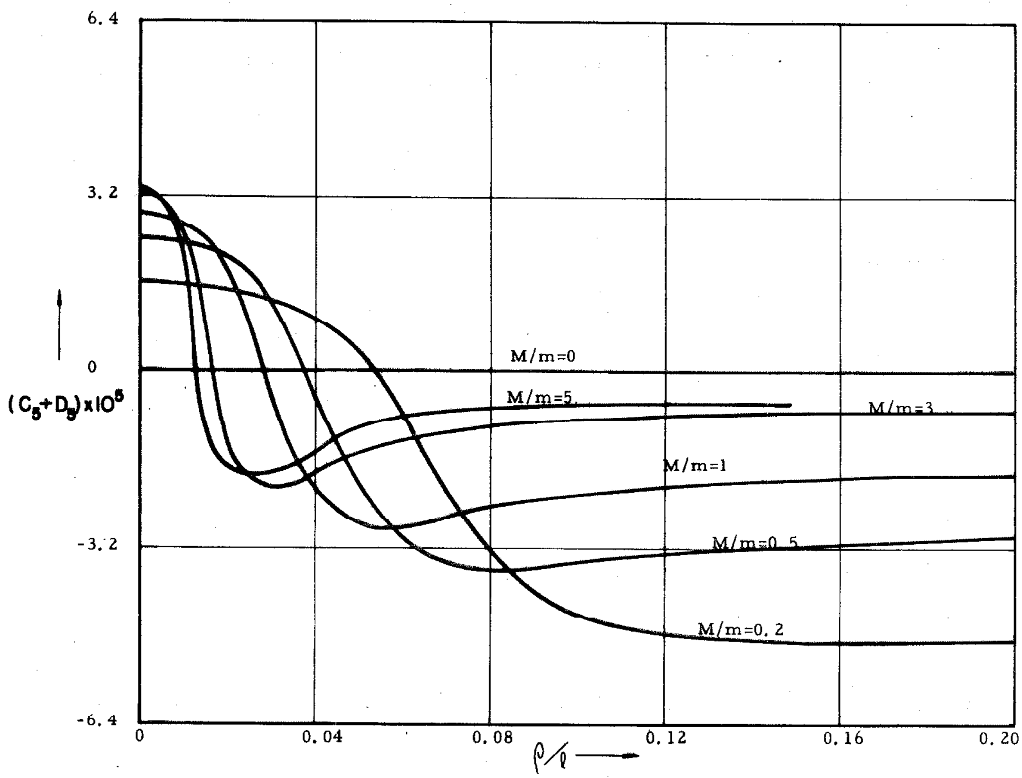
F-5-b(1). FIXED-FIXED, $\ell_1 = \ell/3$ - FIFTH MODE - MODE SHAPE FACTOR



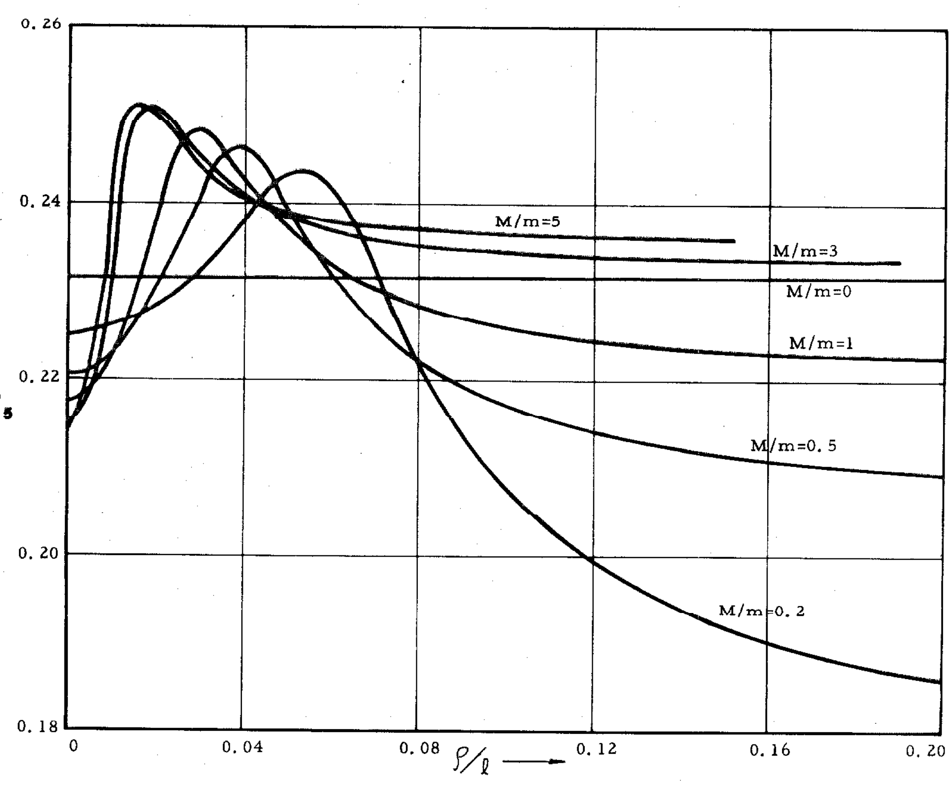
F-5-b(2). FIXED-FIXED, $\ell_1 = \ell/3$ - FIFTH MODE - MODE SHAPE FACTOR



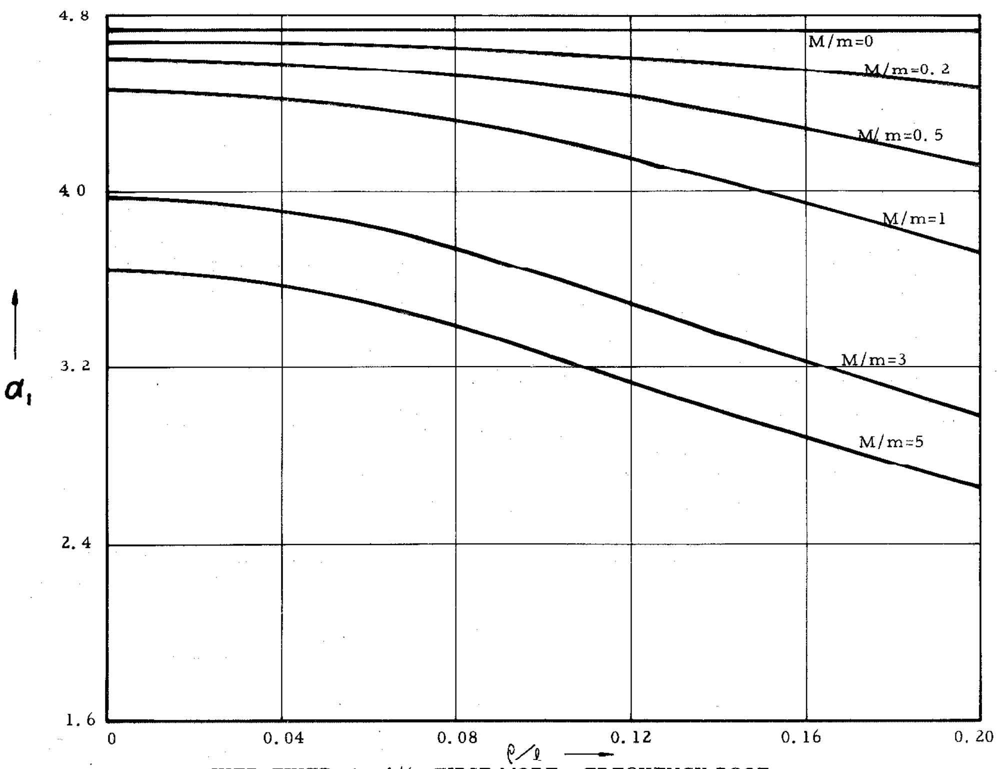
F-5-b(3). FIXED-FIXED, $\ell_1 = \ell/3$ - FIFTH MODE - MODE SHAPE FACTOR



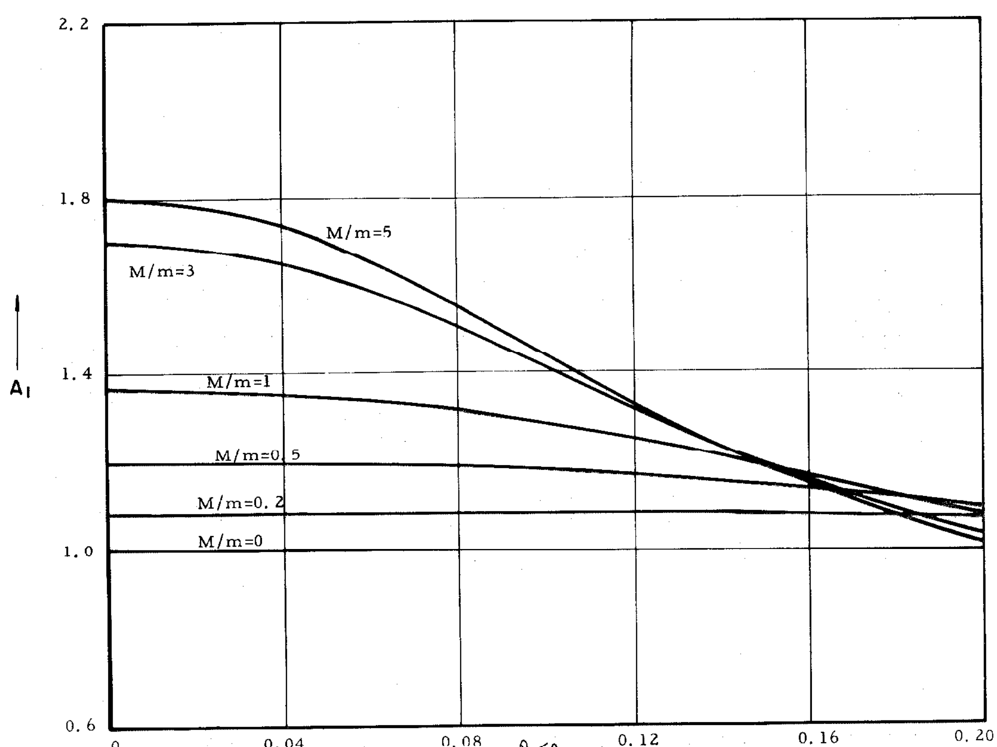
F-5-b(4). FIXED-FIXED, $\ell_1 = \ell/3$ - FIFTH MODE - MODE SHAPE FACTOR



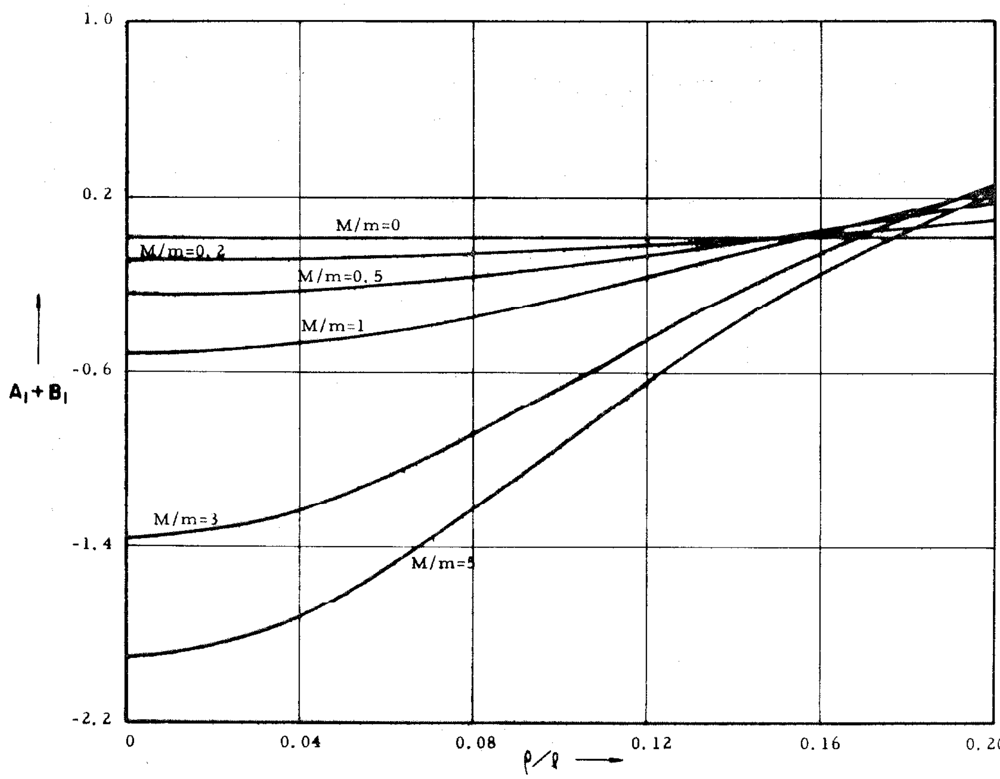
F-5-c. FIXED-FIXED, $\ell_1 = \ell/3$ - FIFTH MODE - MODE PARTICIPATION FACTOR



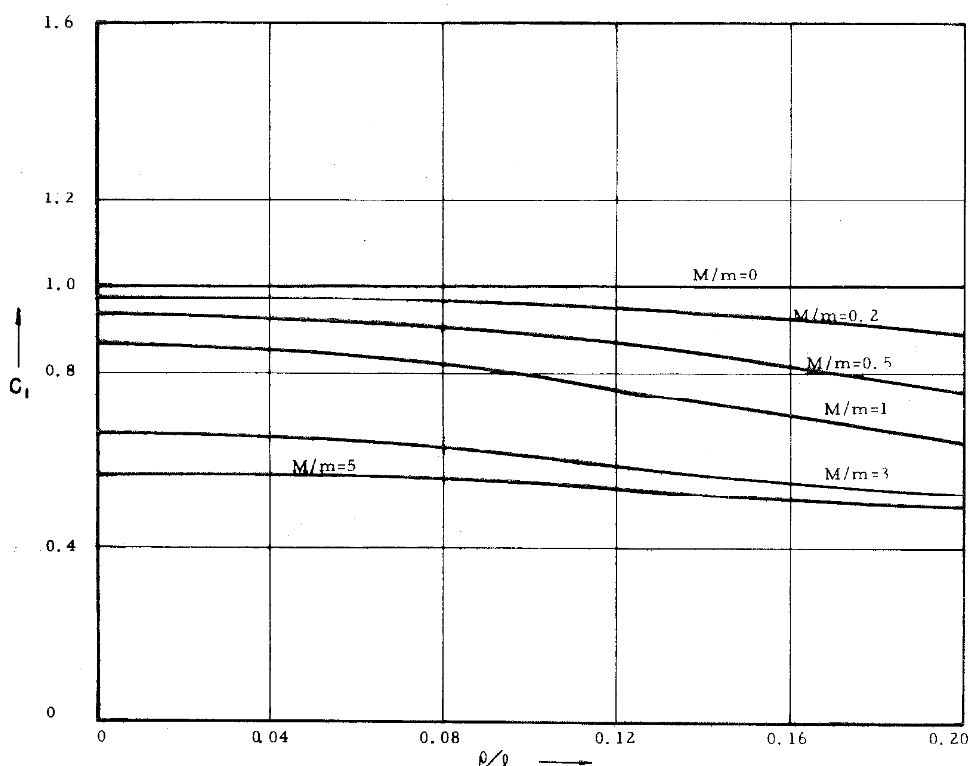
G-1-a. FIXED-FIXED, $\ell_1 = \ell/6$ - FIRST MODE - FREQUENCY ROOT



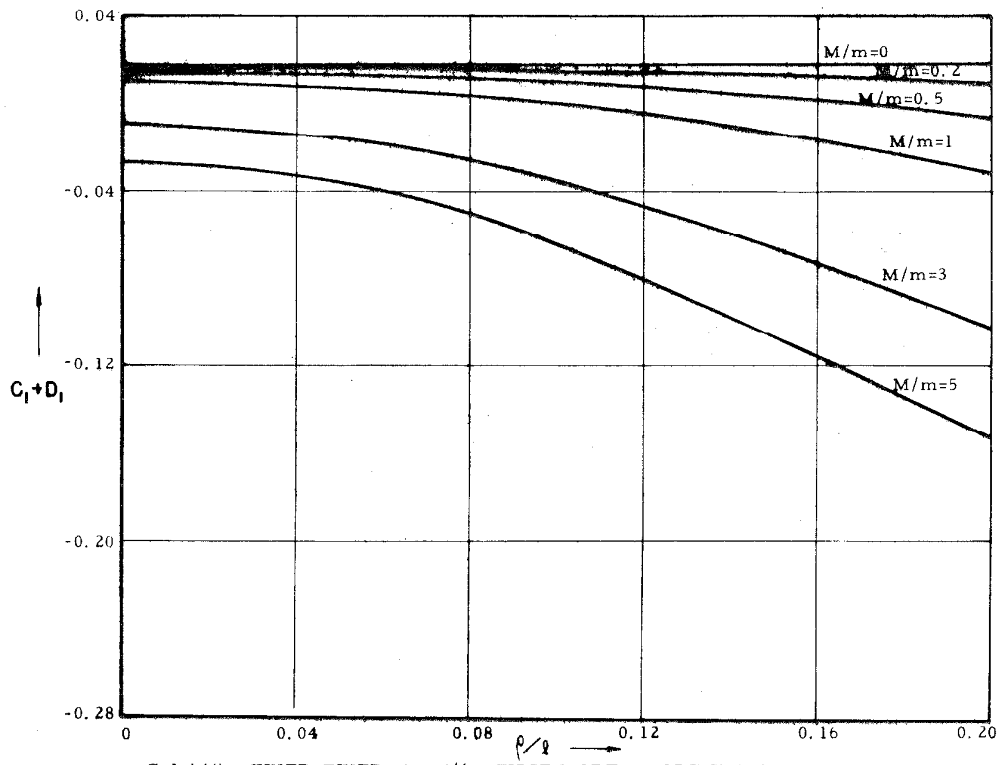
G-1-b(1). FIXED- FIXED, $\ell_1 = \ell/6$ - FIRST MODE - MODE SHAPE FACTOR



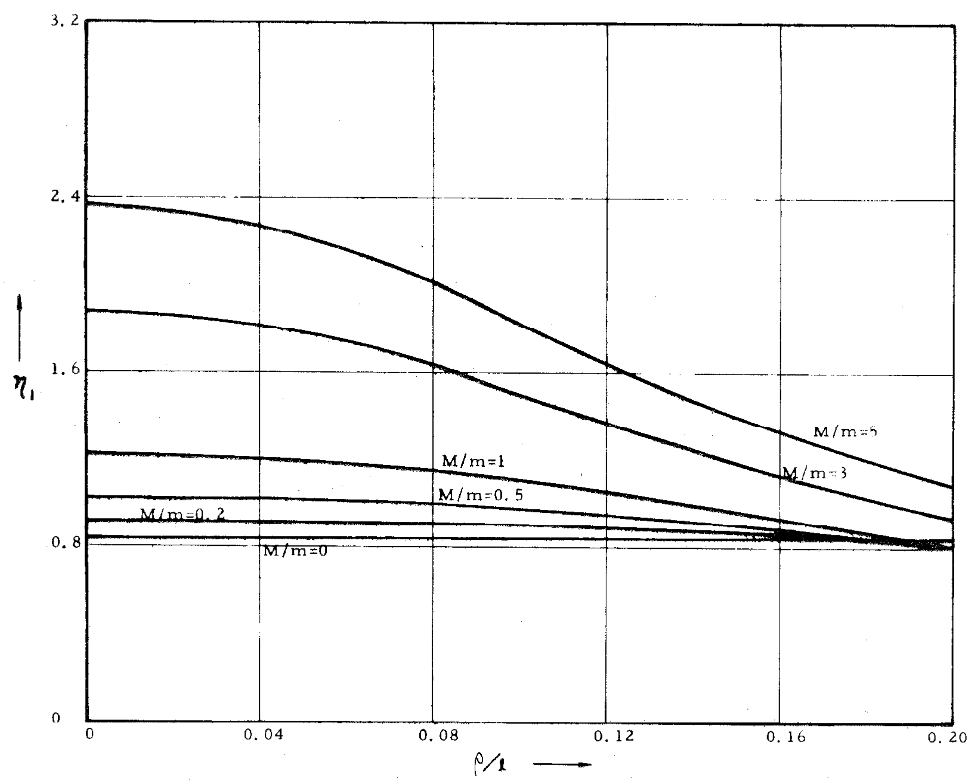
G-1-b(2). FIXED-FIXED, $\ell_1 = \ell/6$ - FIRST MODE - MODE SHAPE FACTOR



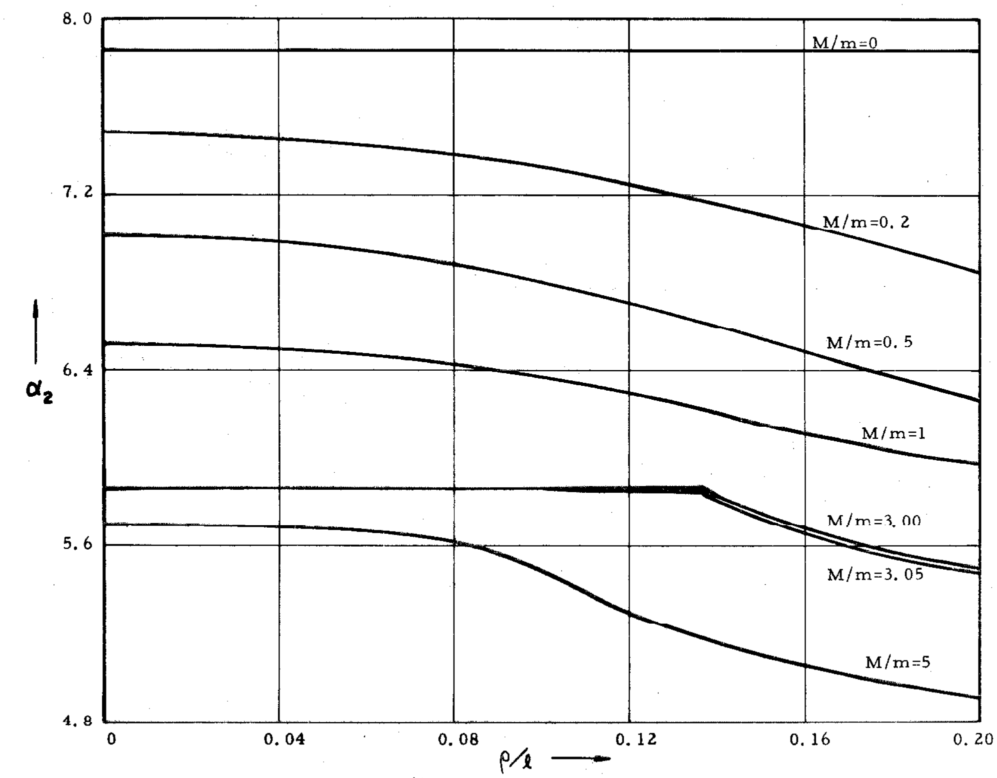
G-1-b(3). FIXED-FIXED, $\ell_1 = \ell/6$ - FIRST MODE - MODE SHAPE FACTOR



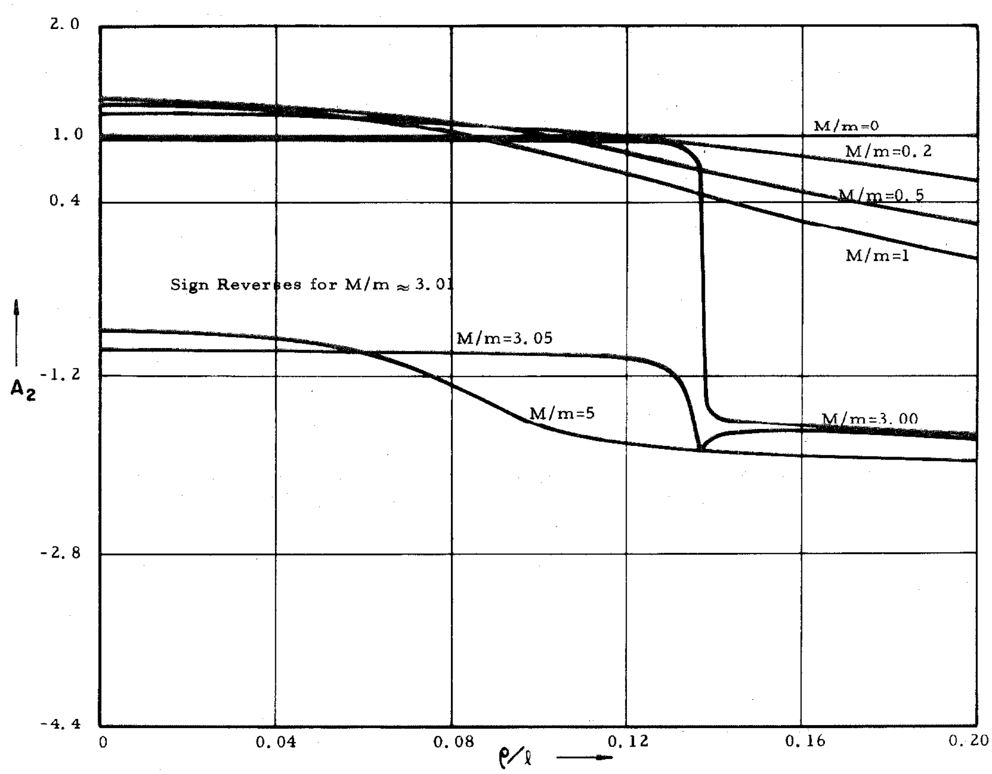
G-1-b(4). FIXED-FIXED, $\ell_1 = \ell/6$ - FIRST MODE - MODE SHAPE FACTOR



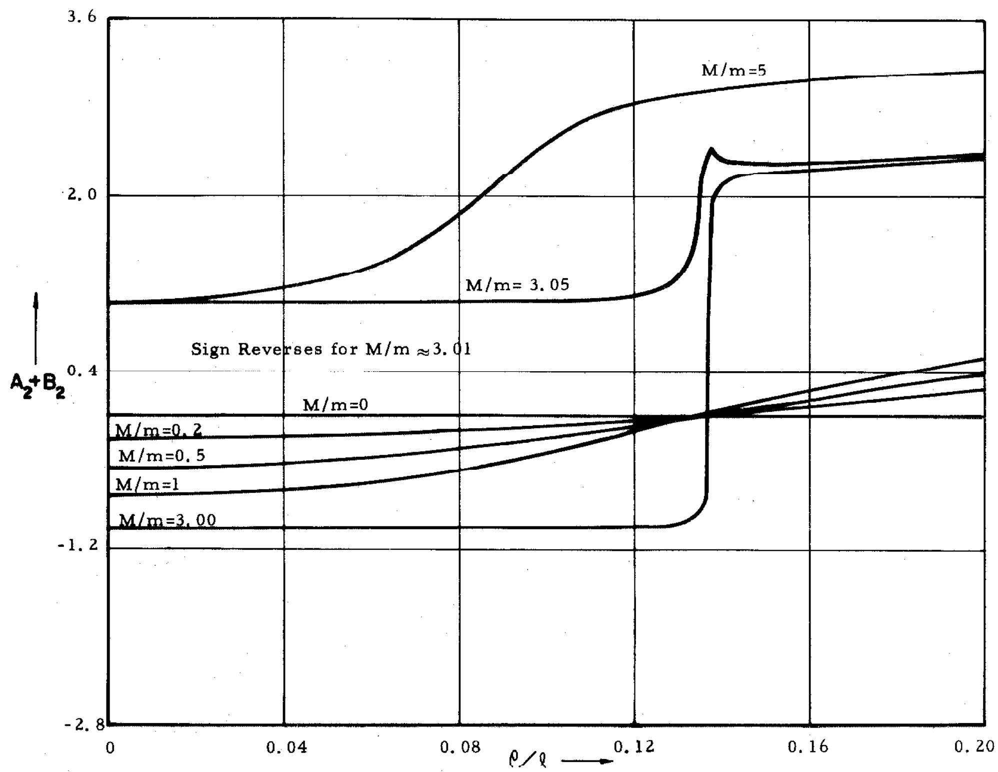
G-1-c. FIXED-FIXED, $\ell_1 = \ell/6$ - FIRST MODE - MODE PARTICIPATION FACTOR



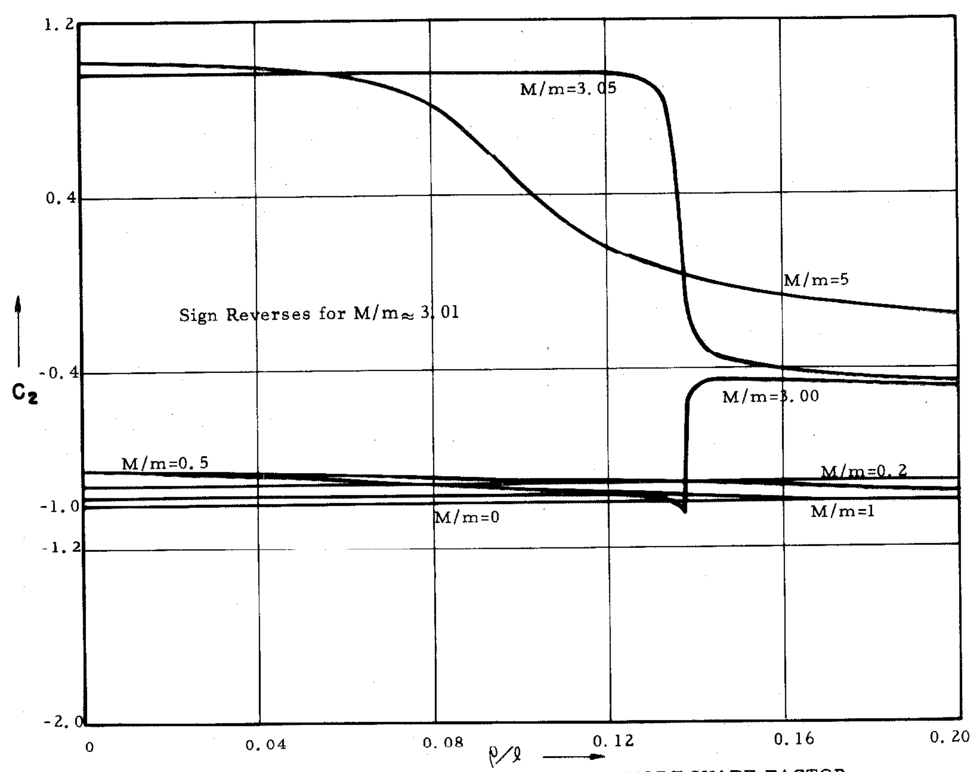
G-2-a. FIXED-FIXED, $\ell_1 = \ell/6$ - SECOND MODE - FREQUENCY ROOT



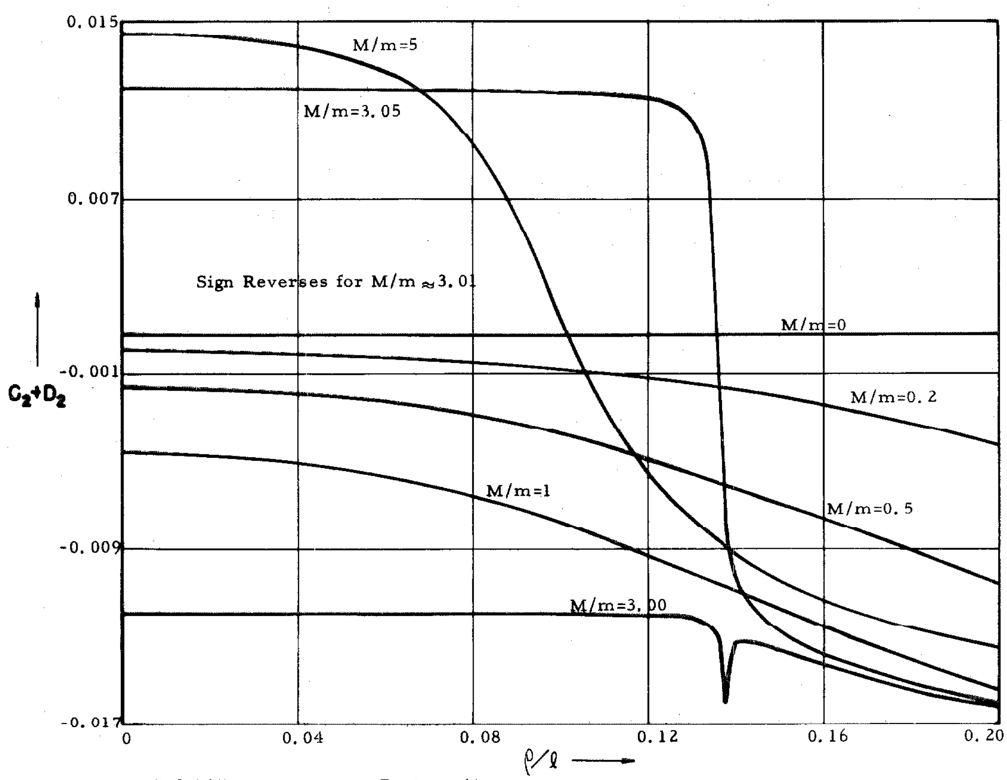
G-2-b(1). FIXED-FIXED, $\ell_1 = \ell/6$ - SECOND MODE - MODE SHAPE FACTOR



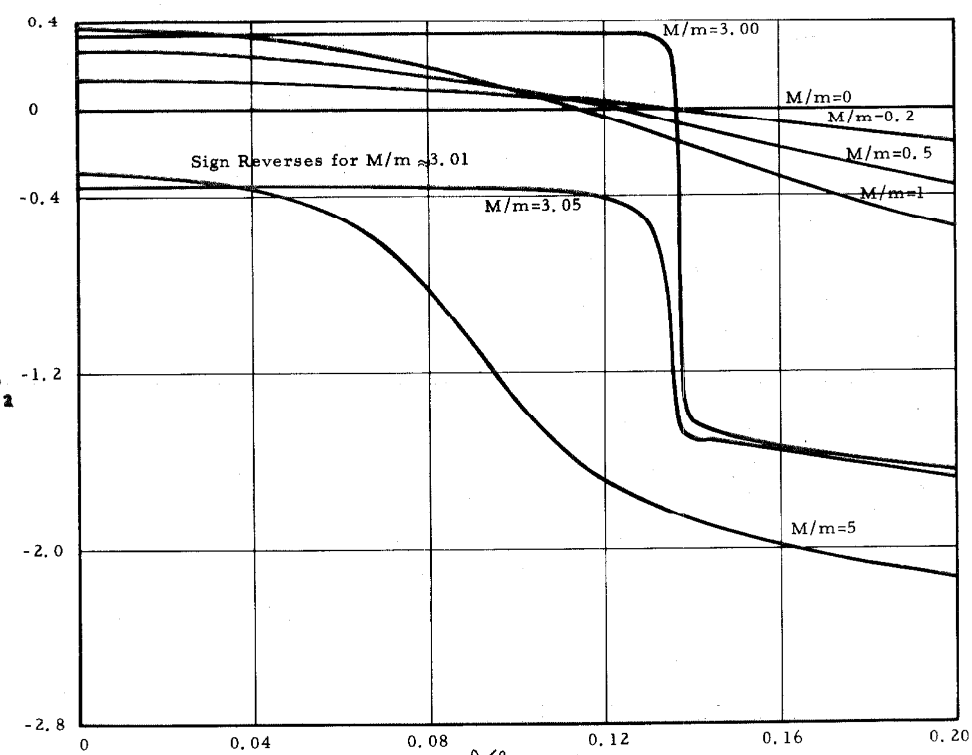
G-2-b(2). FIXED-FIXED, $\ell_1 = \ell/6$ - SECOND MODE - MODE SHAPE FACTOR



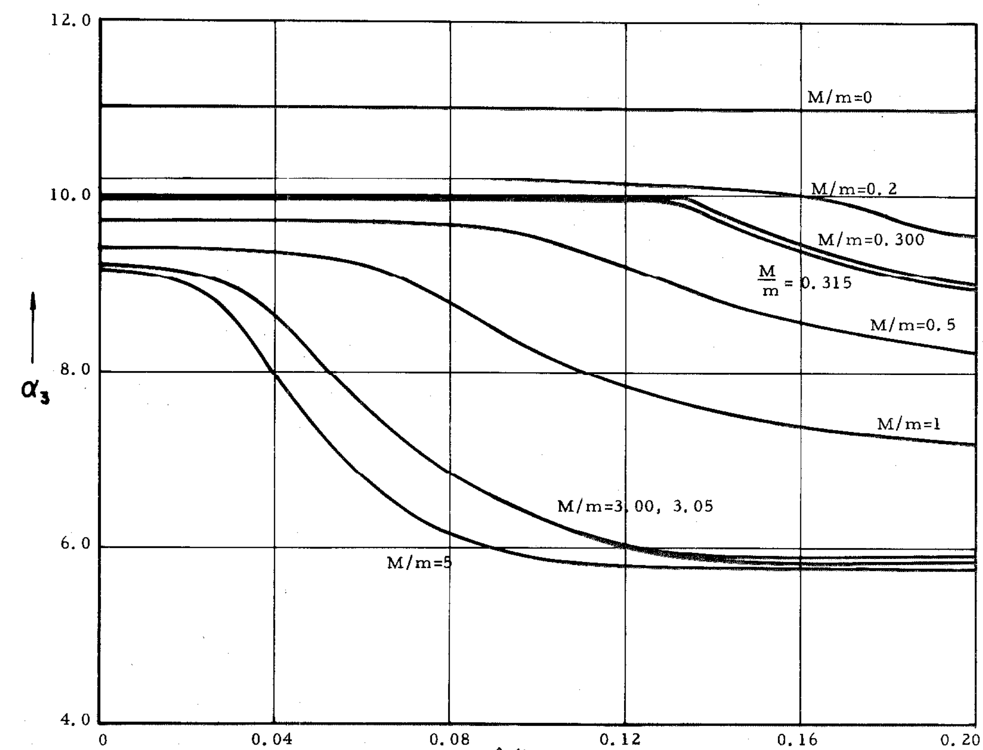
G-2-b(3). FIXED-FIXED, $\ell_1 = \ell/6$ - SECOND MODE - MODE SHAPE FACTOR



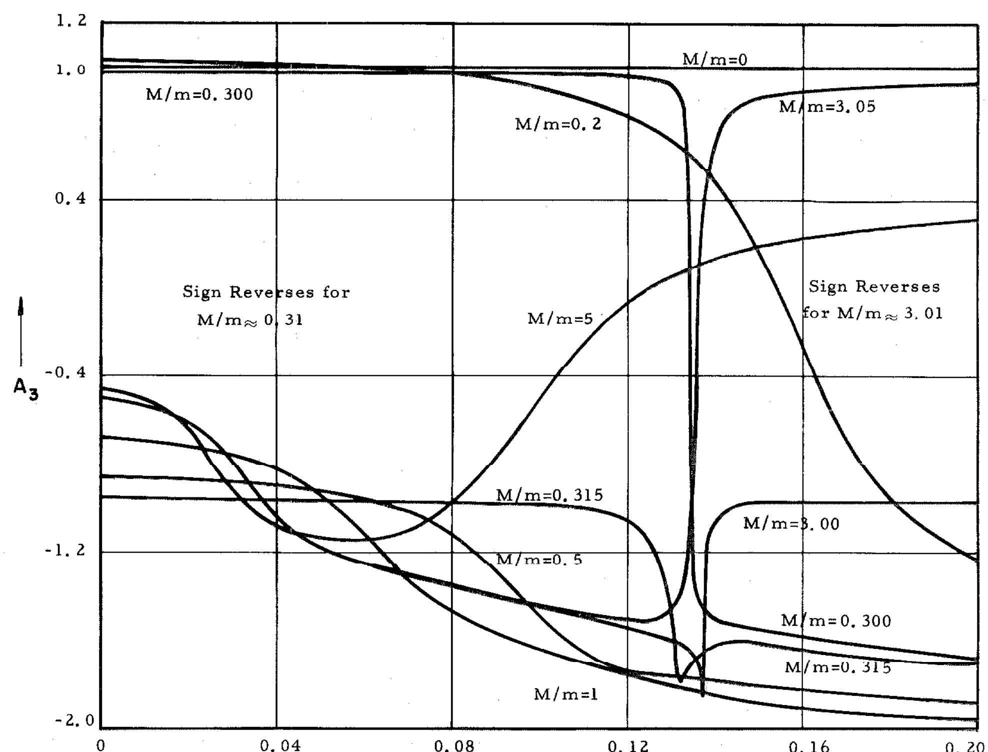
G-2-b(4). FIXED-FIXED, $\ell_1 = \ell/6$ - SECOND MODE - MODE SHAPE FACTOR



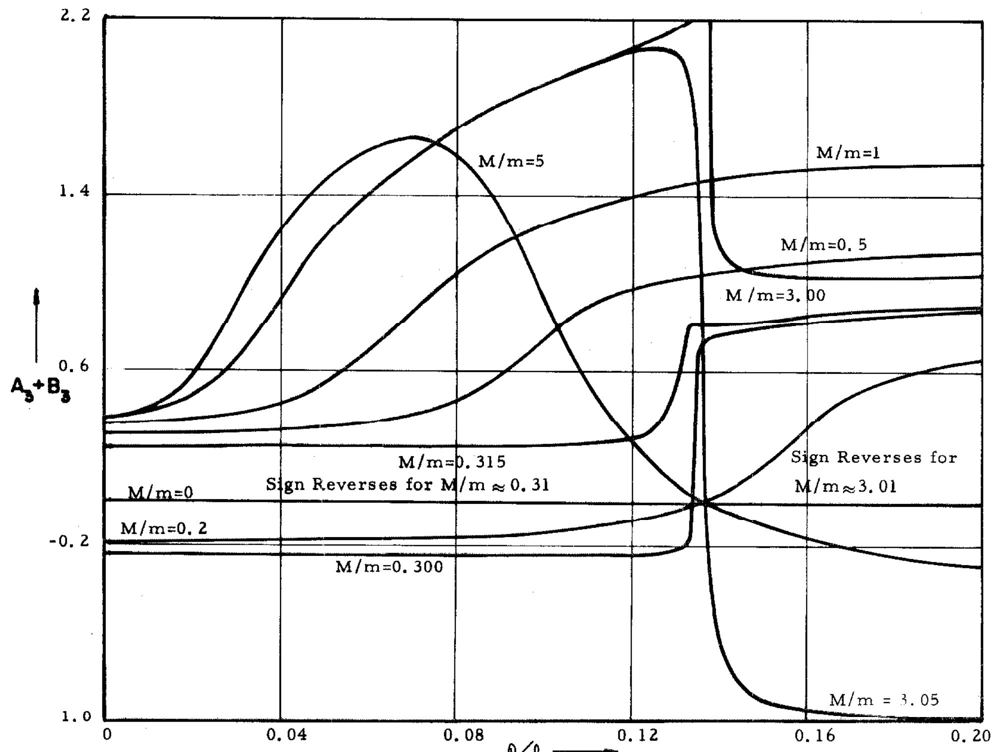
G-2-c. FIXED-FIXED, $\ell_1 = \ell/6$ - SECOND MODE - MODE PARTICIPATION FACTOR



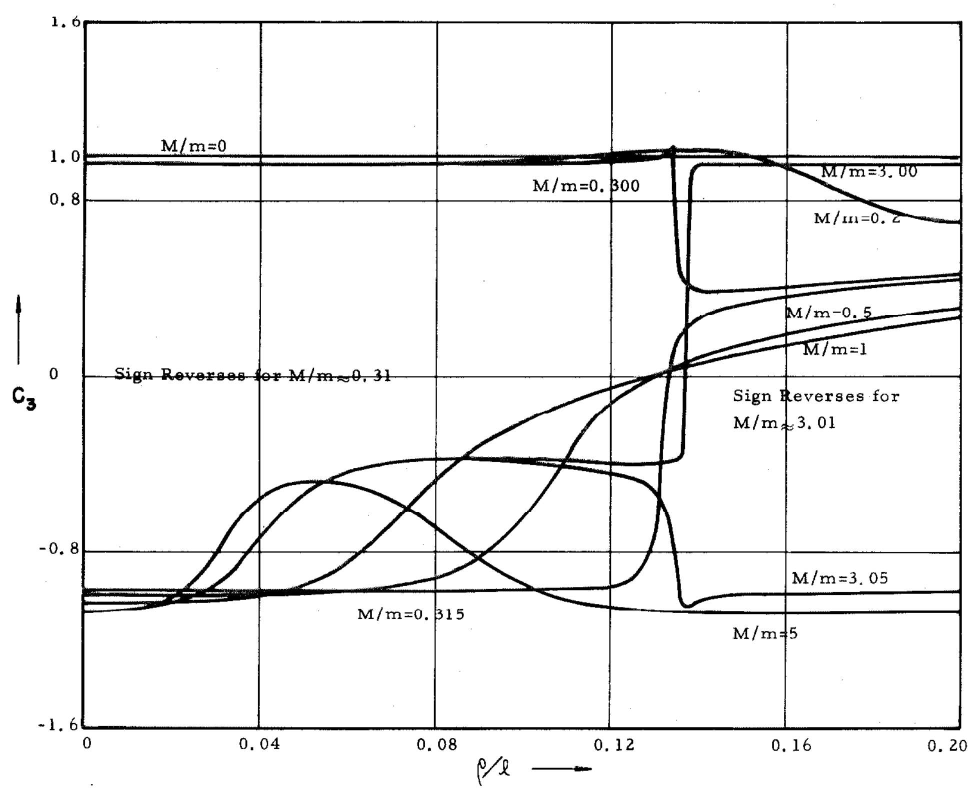
G-3-a. FIXED-FIXED, $\ell_1 = \ell/6$ - THIRD MODE - FREQUENCY ROOT



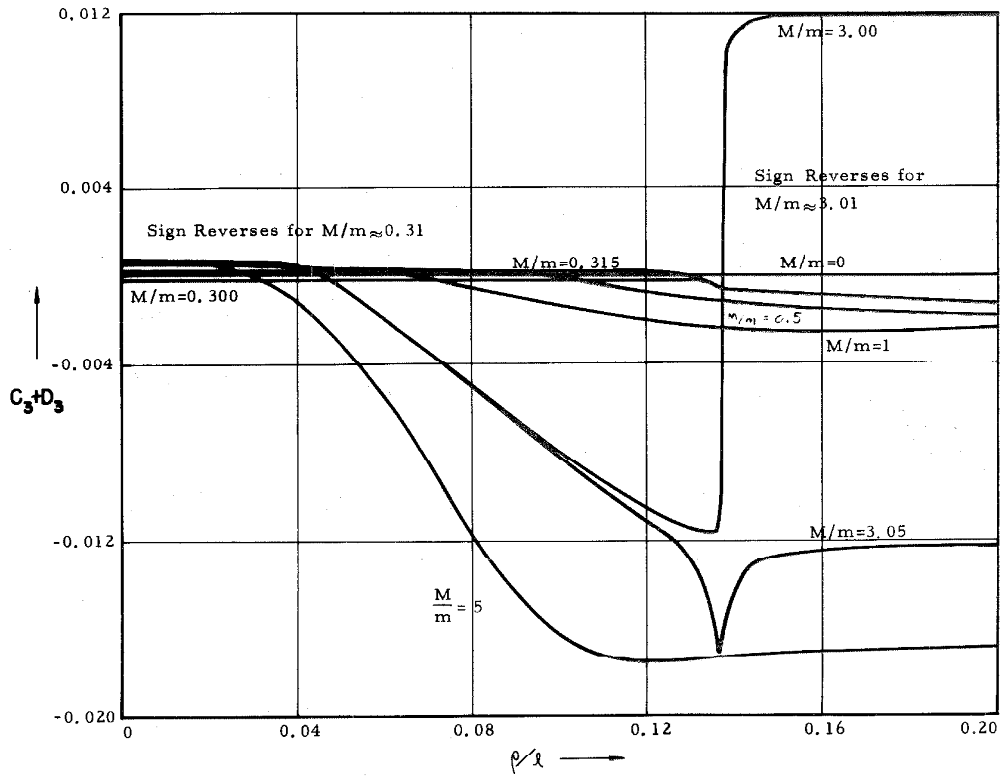
G-3-b(1). FIXED-FIXED, $\ell_1 = \ell/6$ - THIRD MODE - MODE SHAPE FACTOR



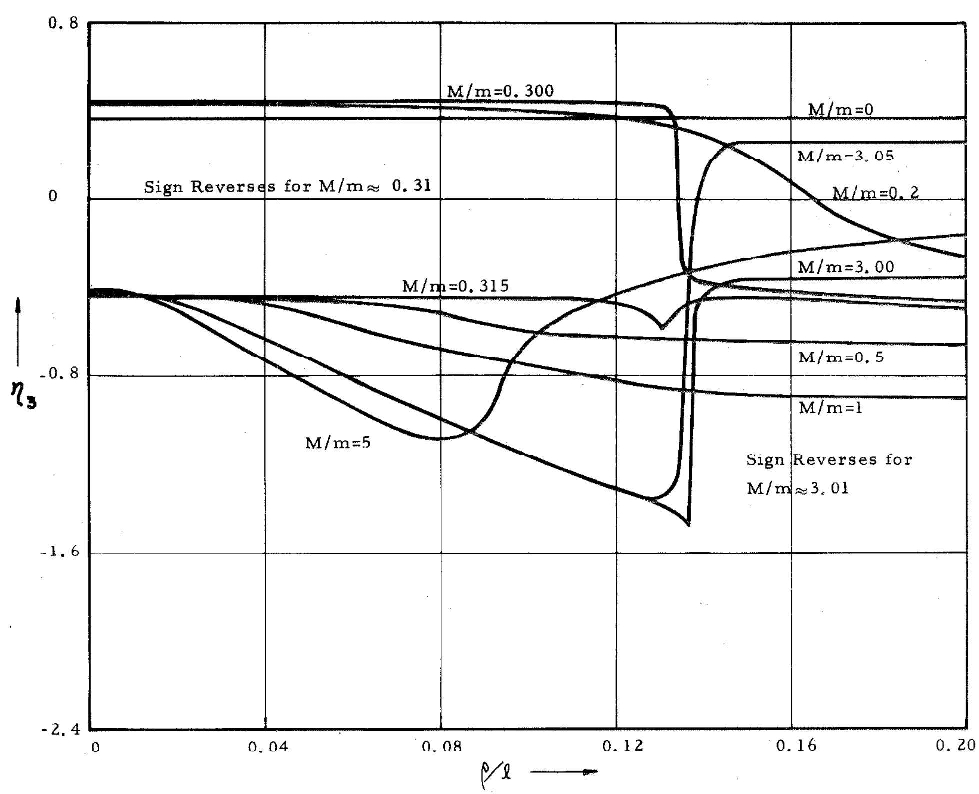
G-3-b(2). FIXED-FIXED, $\ell_1 = \ell/6$ - THIRD MODE - MODE SHAPE FACTOR



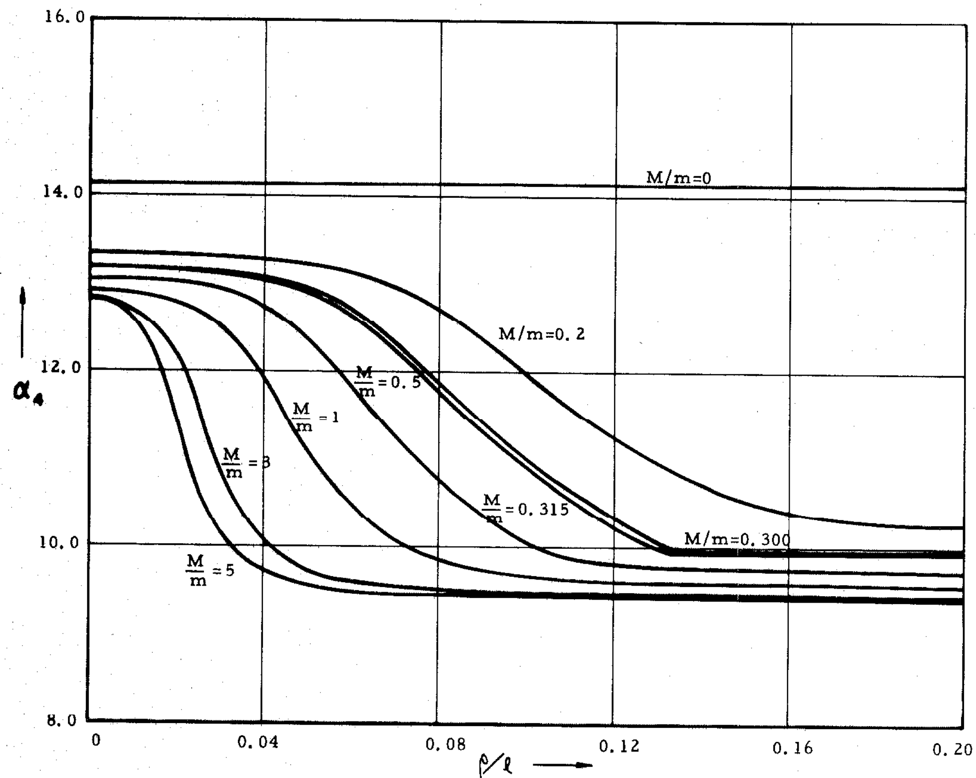
G-3-b(3). FIXED-FIXED, $\ell_1 = \ell/6$ - THIRD MODE - MODE SHAPE FACTOR



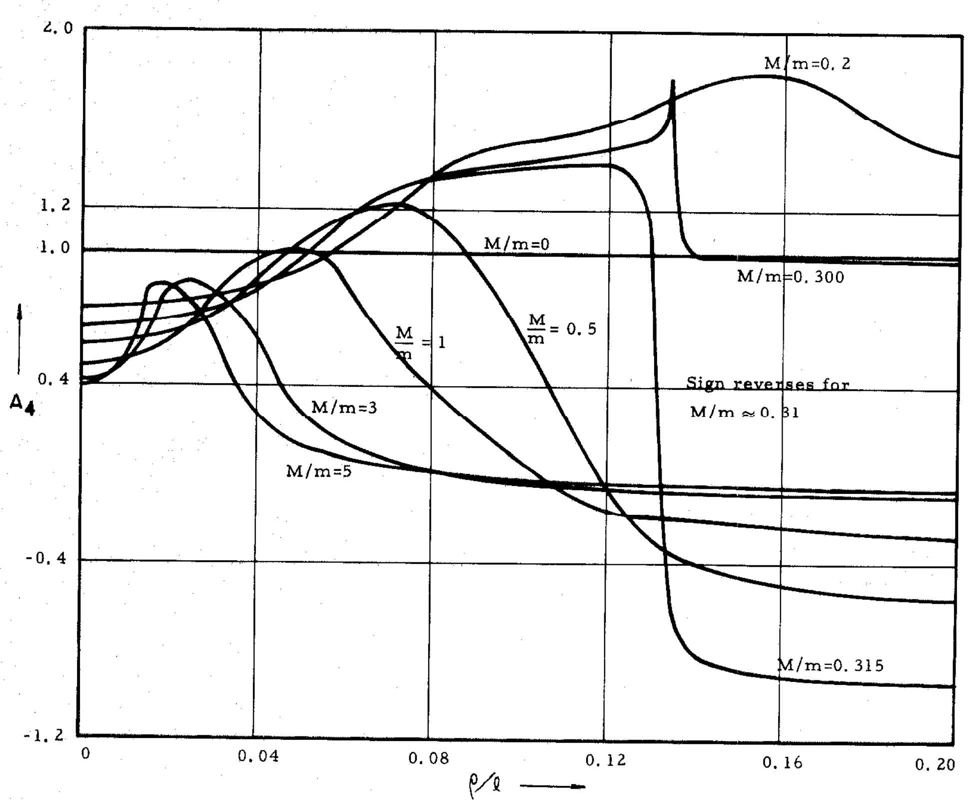
G-3-b(4). FIXED-FIXED, $\ell_1 = \ell/6$ - THIRD MODE - MODE SHAPE FACTOR



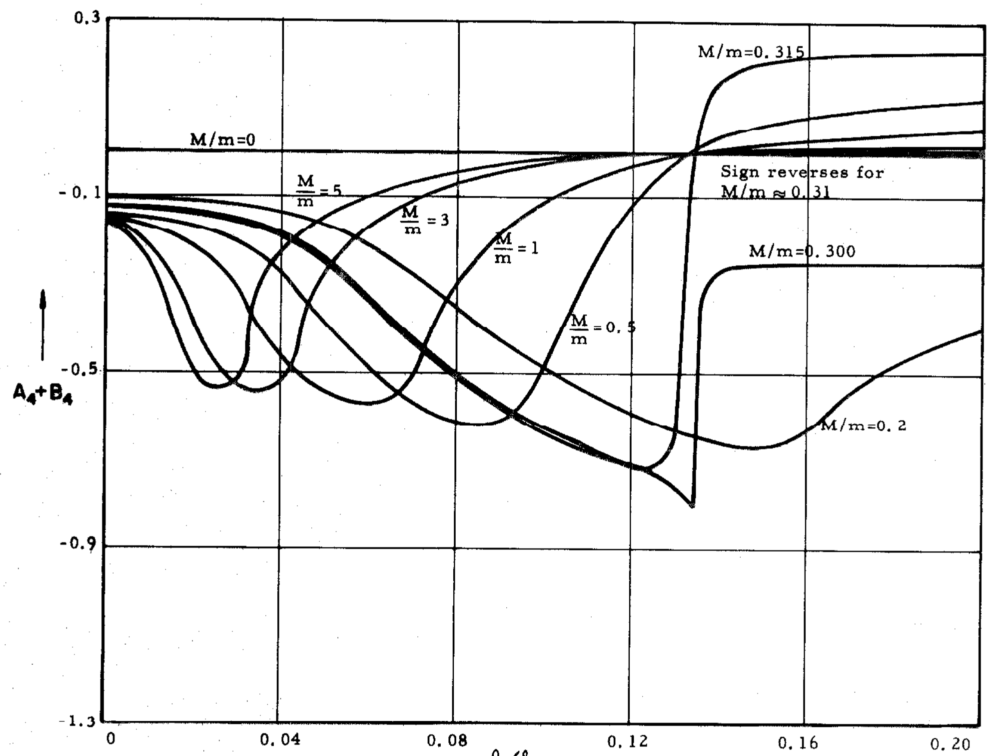
G-3-c. FIXED-FIXED, $\ell_1 = \ell/6$ - THIRD MODE - MODE PARTICIPATION FACTOR



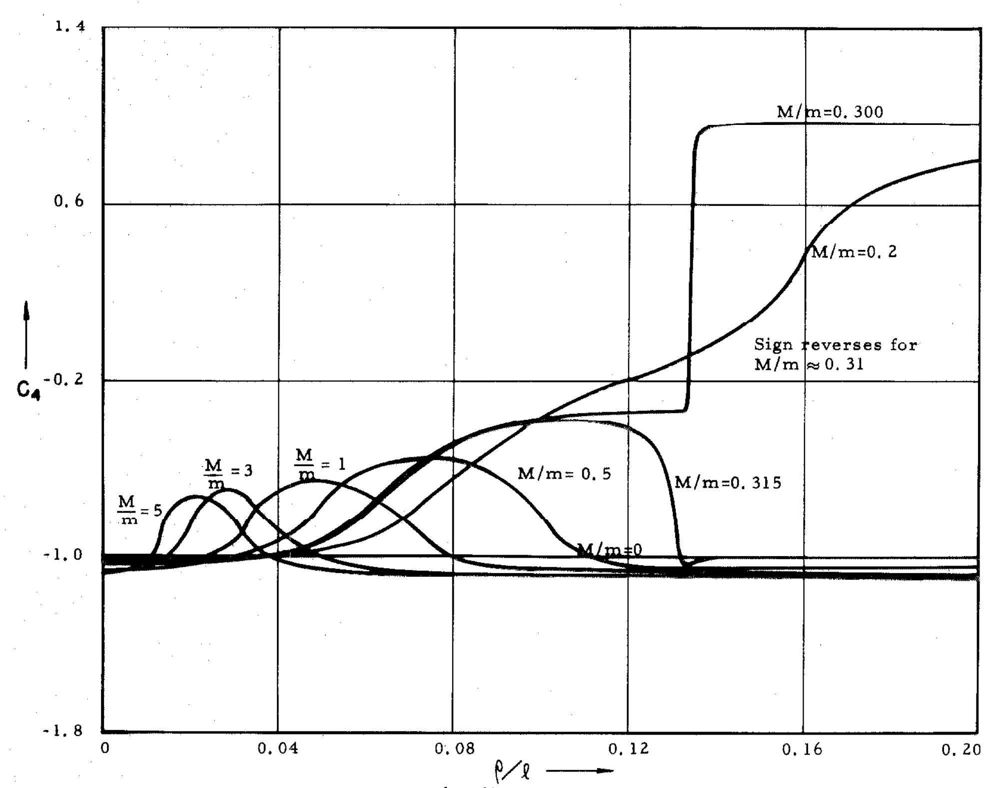
G-4-a. FIXED-FIXED, $\ell_1 = \ell/6$ - FOURTH MODE - FREQUENCY ROOT



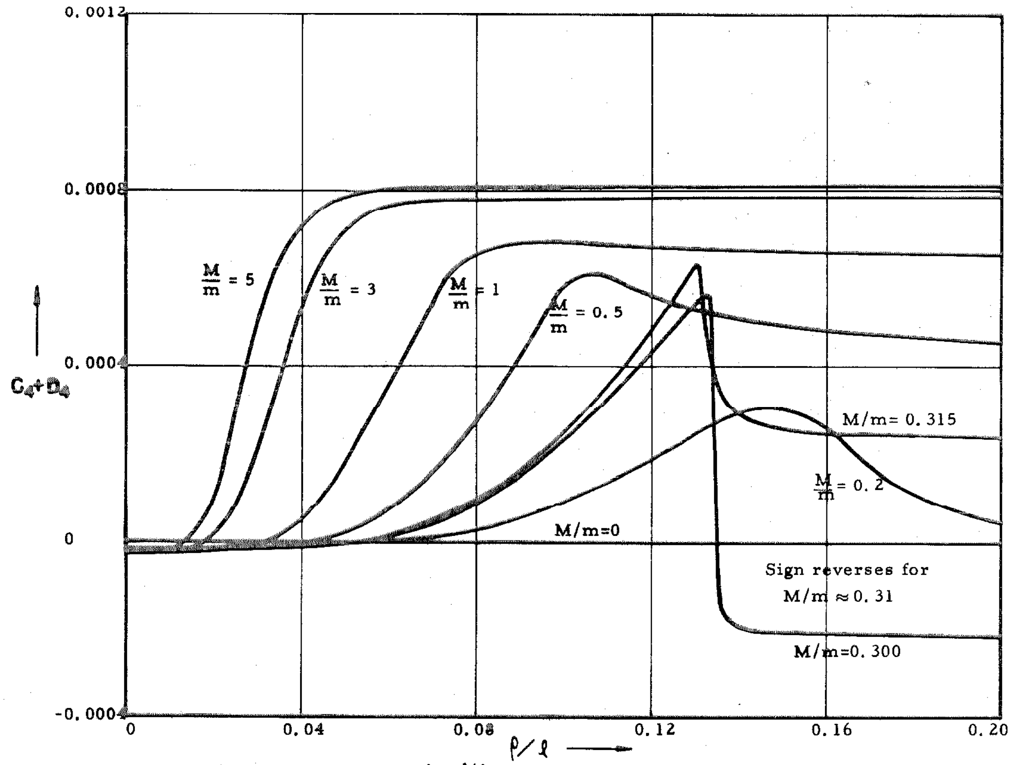
G-4-b(1). FIXED-FIXED, $\ell_1 = \ell/6$ - FOURTH MODE - MODE SHAPE FACTOR



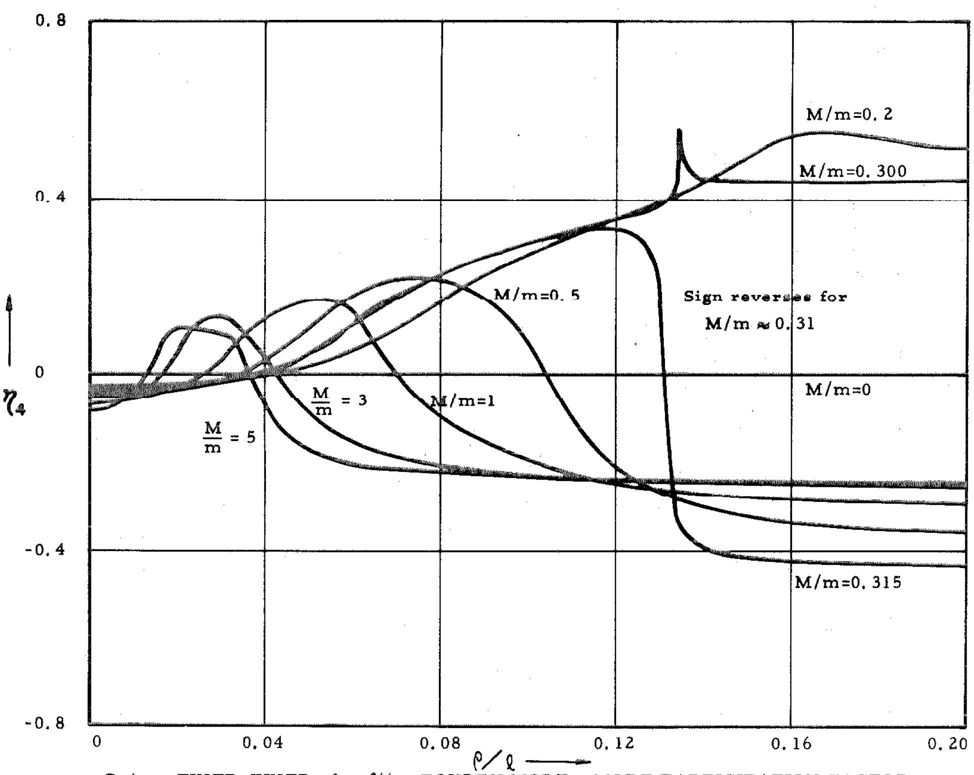
G-4-b(2). FIXED-FIXED, $\ell_1 = \ell/6$ - FOURTH MODE - MODE SHAPE FACTOR



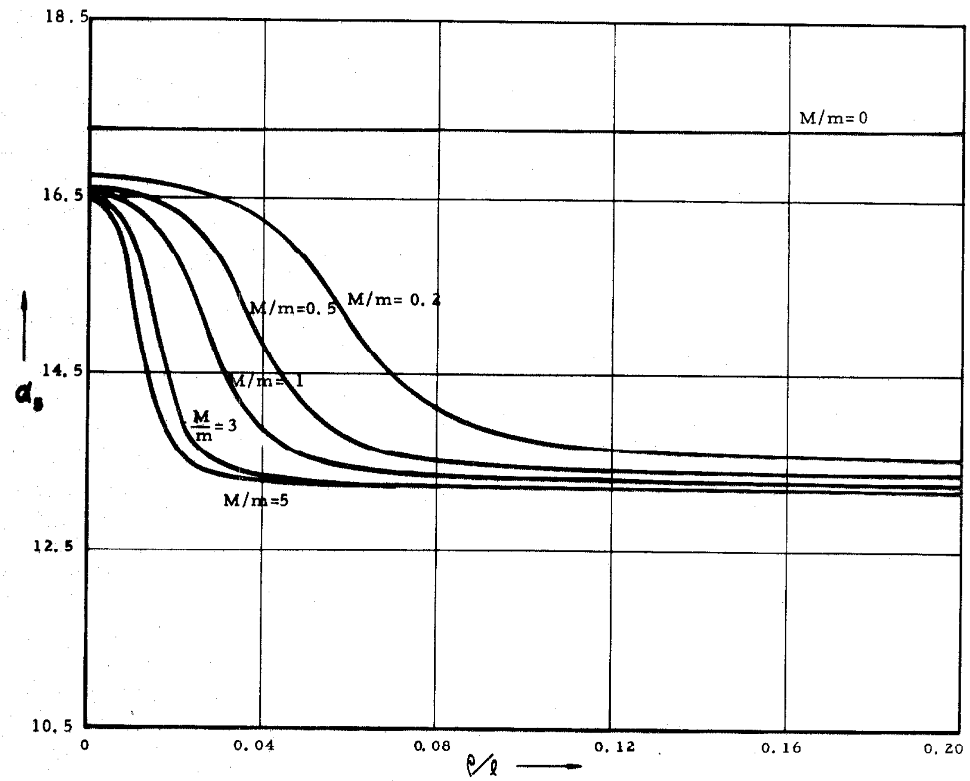
G-4-b(3). FIXED-FIXED, $\ell_1 = \ell/6$ - FOURTH MODE - MODE SHAPE FACTOR



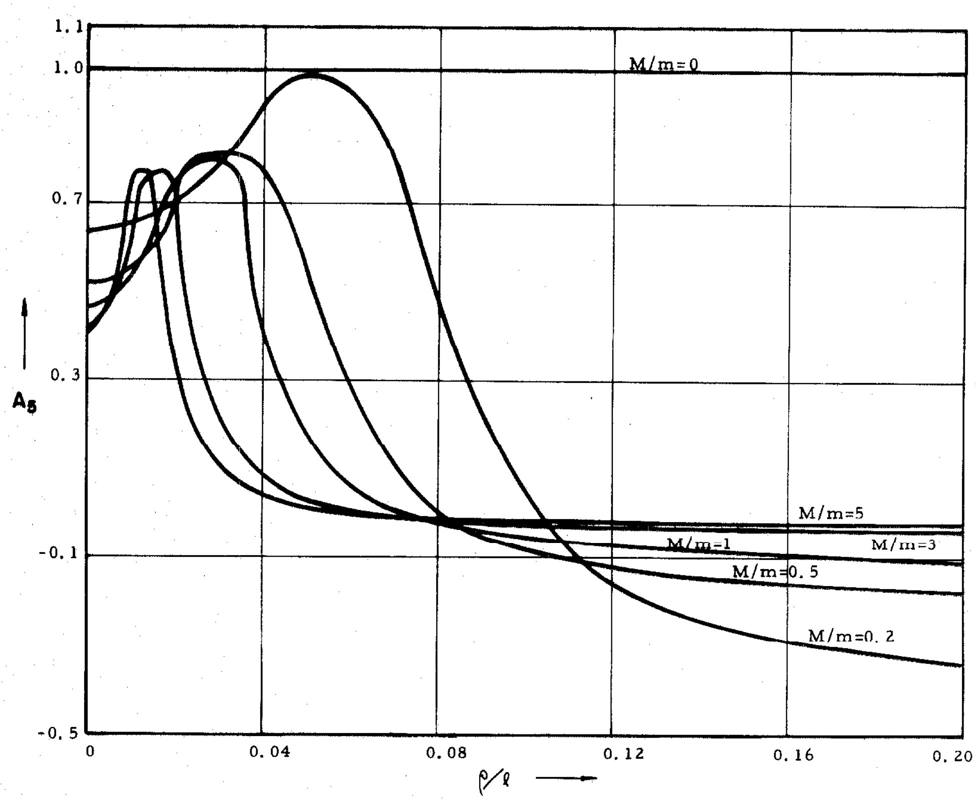
G-4-b(4). FIXED-FIXED, $\ell_1 = \ell/6$ - FOURTH MODE - MODE SHAPE FACTOR



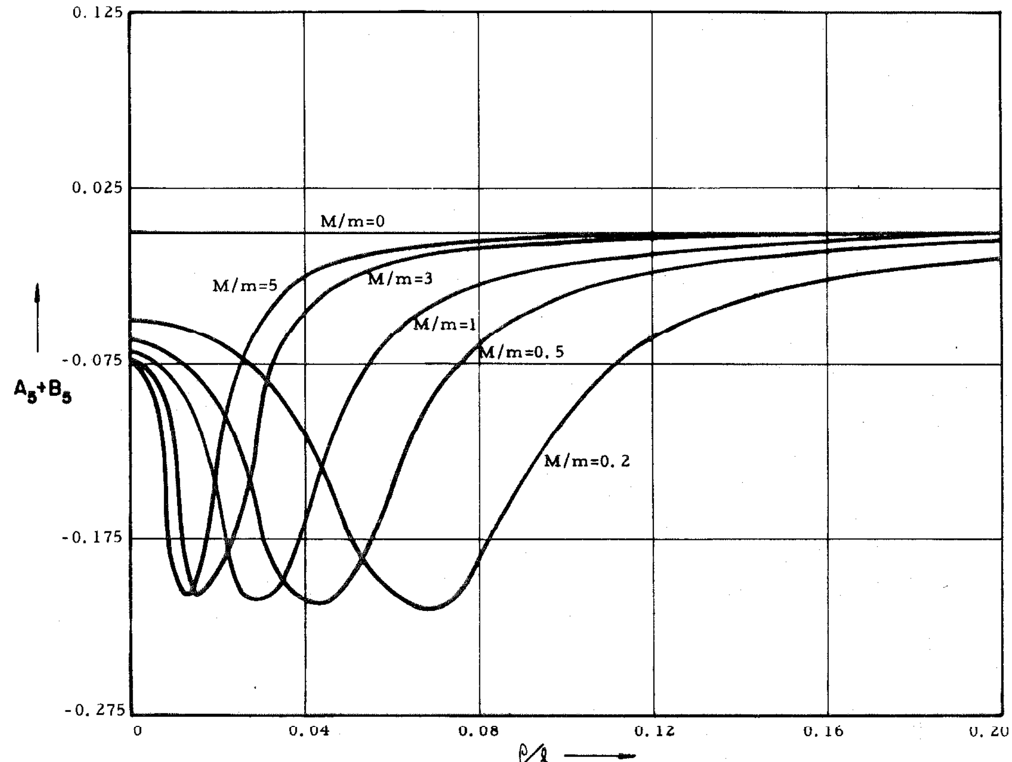
G-4-c. FIXED-FIXED, $\ell_1 = \ell/6$ - FOURTH MODE - MODE PARTICIPATION FACTOR



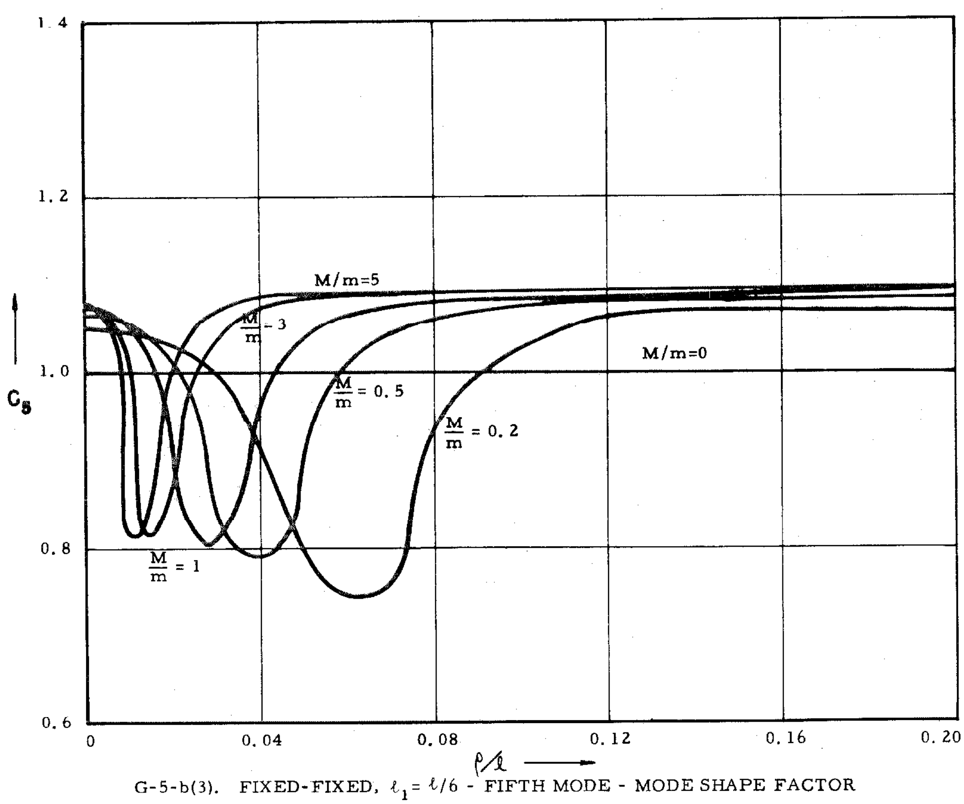
G-5-a. FIXED-FIXED, $\ell_1 = \ell/6$ - FIFTH MODE - FREQUENCY ROOT



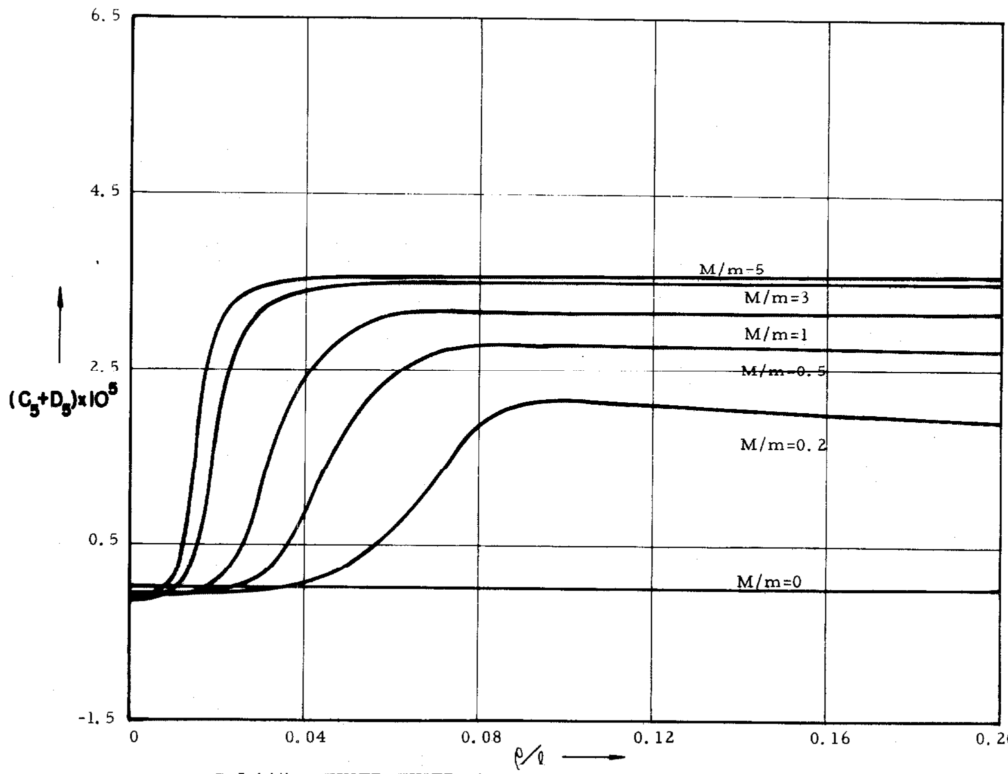
G-5-b(1). FIXED-FIXED, $\ell_1 = \ell/6$ - FIFTH MODE - MODE SHAPE FACTOR



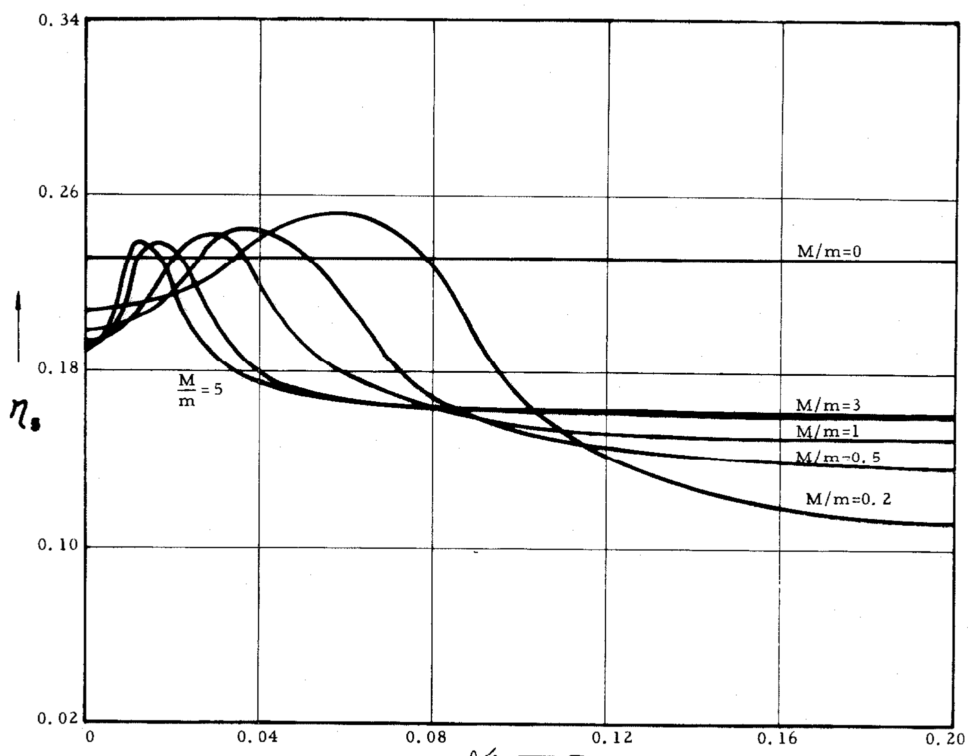
G-5-b(2). FIXED-FIXED, $\ell_1 = \ell/6$ - FIFTH MODE - MODE SHAPE FACTOR



G-5-b(3). FIXED-FIXED, $\ell_1 = \ell/6$ - FIFTH MODE - MODE SHAPE FACTOR



G-5-b(4). FIXED-FIXED, $\ell_1 = \ell/6$ - FIFTH MODE - MODE SHAPE FACTOR



G-5-c. FIXED-FIXED, $\ell_1 = \ell/6$ - FIFTH MODE - MODE PARTICIPATION FACTOR